Homework 5

Directional Derivative & Gradient, Maxima/Minima/Saddle Points, Second Derivative Test

Due: Wed Sept 27
Reading: Sections 11.5–11.7

Please label with MAT 267H, HW5, & your name; write up the problems neatly & in numerical order; box short answers; & write (2) — if problem (2) is not attempted. No need to restate problem. Derive your answers!

When writing BY HAND, use $\vec{v}$ for vectors.

(1) Suppose the temperature in a gas is given by

$$T(x, y, z) = \exp \left\{ - \left( x^2 + y^2 + \frac{|z|}{4} \right) \right\}$$

(a) Calculate $\nabla T$ for $z \geq 0$.
(b) In which direction is $T$ decreasing the fastest at $(1, 1, 1)$?
(c) What is the directional derivative of $T$ along $u = (1, -1, 2)$ at $(1, 1, 1)$?

Ans: (a) $\nabla T = (-2x, -2y, -1/4)\ T$, (b) $-\nabla T(1, 1, 1) = (2, 2, 1/4)\ \exp\{-9/4\}$,
(c) $D_u T(1, 1, 1) = (1/\sqrt{6})(1, -1, 2) \cdot (-2, -2, -1/4)\ \exp\{-9/4\} = -\exp\{-9/4\} / (2\sqrt{6})$

(2) Show that $u(x, t) = f(x - ct) + g(x + ct)$ satisfies the wave equation $u_{tt} - c^2 u_{xx} = 0$, demonstrating that there are two waves: one propagating to the right with velocity $c$ and one to the left with velocity $-c$. Ans: $u_{xx} = f''(x - ct) + g''(x + ct), \ u_{tt} = c^2 f''(x - ct) + c^2 g''(x + ct)$

(3) Show that $u(x, y) = \ln(\sqrt{x^2 + y^2}) = \ln(r)$ satisfies the 2D Laplace Equation $\nabla^2 u = u_{xx} + u_{yy} = 0$. Hint: Write $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$. Ans: $u_x = x/(x^2 + y^2), \ u_{xx} = 1/(x^2 + y^2) - 2x^2/(x^2 + y^2)^2, \ u_y = y/(x^2 + y^2), \ u_{yy} = 1/(x^2 + y^2) - 2y^2/(x^2 + y^2)^2$

For problems (4)–(8), find and classify the critical points, using the second derivative test:
(4) \( f(x, y) = x^3 + y^3 \)

*Hint:* Show that \( \Delta(0, 0) = 0 \) (second derivative test does not apply)—have to graph to determine type of critical point. *Ans:* \( f_x = 3x^2 = 0 = f_y = 3y^2 \) implies only \((0, 0)\) is a critical point; \( f_{xx} = 6x, f_{yy} = 6y, f_{xy} = 0, \Delta(0, 0) = 0. \) Graphically \( f \) has a saddle at \((0, 0)\).

(5) \( f(x, y) = x \sin(y) \)

*Ans:* \( f_x = \sin(y) = 0 = f_y = x \cos(y) \) implies critical points are \((0, n\pi), n \text{ an integer } 0, \pm 1, \pm 2, \ldots; f_{xx} = 0, f_{yy} = -x \sin(y), f_{xy} = \cos(y), \Delta = -\cos^2(y); \) at the critical points, \( \Delta = -1, \) saddle points.

(6) \( f(x, y) = x^2y^2 - x \)

*Hint:* Show there are no stationary points. *Ans:* \( f_x = 2xy^2 - 1 = 0 = f_y = 2x^2y \) has no solutions.

(7) \( f(x, y) = xe^y - e^x \)

*Hint:* \((0, 0)\) is a saddle point. *Ans:* \( f_x = e^y - e^x = 0 = f_y = xe^y \) implies \((0, 0)\) is the only critical point; \( f_{xx} = -e^x, f_{yy} = xe^y, f_{xy} = e^y, \Delta = -xe^{x+y} - e^{2y}, \Delta(0, 0) = -1. \)

(8) \( f(x, y) = \sin(x) - \cos(y) \)

*Hint:* \((\pi/2 + n\pi, m\pi), m \text{ and } n \text{ integers, minimum when } f = -2, \) saddle when \( f = 0, \) maximum when \( f = 2: \) show this using \( \Delta \) and identify values of \( m \) and \( n \) for min, max, saddle. *Ans:* \( f_x = \cos(x) = 0 = f_y = \sin(y) \) implies \((\pi/2 + n\pi, m\pi)\) are the critical points; \( f_{xx} = -\sin(x), f_{yy} = \cos(y), f_{xy} = 0, \Delta = -\sin(x) \cos(y). \)

<table>
<thead>
<tr>
<th>critical points</th>
<th>( \Delta )</th>
<th>( f_{xx} )</th>
<th>type</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\pi/2, 0) )</td>
<td>-1</td>
<td>saddle</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( (3\pi/2, \pi) )</td>
<td>-1</td>
<td>saddle</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( (\pi/2, \pi) )</td>
<td>+1</td>
<td>-1</td>
<td>max</td>
<td>2</td>
</tr>
<tr>
<td>( (3\pi/2, 0) )</td>
<td>+1</td>
<td>+1</td>
<td>min</td>
<td>-2</td>
</tr>
</tbody>
</table>

Table 1: Basic critical points \((x_0, y_0)\) for \( 0 \leq x_0 < 2\pi \) and \( 0 \leq y_0 < 2\pi. \) For this problem, because of the periodicity of \( \sin() \) and \( \cos(), \) \((x_0, y_0)\) may be replaced by \((x_0 + 2n\pi, y_0 + 2m\pi)\) without changing the type of the critical point or the values of \( f \) and its partial derivatives. Note that only the signs are needed for \( \Delta \) and \( f_{xx}. \)