Objectives

1. Calculate the number of permutations of $n$ objects taken $r$ at a time.
2. Use factorial notation to represent the number of permutations of a set of objects.
3. Calculate the number of combinations of $n$ objects taken $r$ at a time.
4. Apply the theory of permutations and combinations to solve counting problems.

A student, in a moment of frustration, once said to me, “When you first explain a new idea, I understand what you are doing. Then you keep on explaining it until I don’t understand it at all!”* What I believe was frustrating this student is what you also may be dealing with in this chapter. At first we gave you simple, intuitive ways to count using systematic listing and drawing tree diagrams. Then, we went a little more deeply into the idea by introducing the fundamental counting principle. Now we are going to continue our line of thought by giving mathematical-sounding names and notation to the patterns that you have seen earlier.

For example, in Section 13.2, when we constructed keypad patterns in Example 4, you saw the product $10 \times 9 \times 8 \times 7 \times 6$. In Example 5, after seating Louise and her tutor, we used the pattern $8 \times 7 \times 6 \times 5$ to count the ways to fill the remaining four seats. We will now explore these patterns in more detail.

**Permutations**

In both of these problems, we were selecting objects from a set and arranging them in order in a straight line. This notion occurs so often that we give it a name.

---

*I sincerely hope that this doesn’t happen to you in this section.*

---

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A permutation is an ordering of distinct objects in a straight line. If we select r different objects from a set of n objects and arrange them in a straight line, this is called a permutation of n objects taken r at a time. The number of permutations of n objects taken r at a time is denoted by \( P(n, r) \).*

The following diagram will help you to remember the meaning of the notation \( P(n, r) \).

\[
P(n, r) = \frac{n!}{(n-r)!}
\]

For example \( P(5, 3) \) indicates that you are counting permutations (straight-line arrangements) formed by selecting three different objects from a set of five available objects.

**EXAMPLE 1** **Counting Permutations**

a) How many permutations are there of the letters a, b, c, and d? Write the answer using \( P(n, r) \) notation.

b) How many permutations are there of the letters a, b, c, d, e, f, and g if we take the letters three at a time? Write the answer using \( P(n, r) \) notation.

**SOLUTION:**

a) In this problem, we are arranging the letters a, b, c, and d in a straight line without repetition. For example, abcd would be one permutation, and bacd would be another. There are four letters that we can use for the first position, three for the second, and so on. Figure 13.11 is a slot diagram for this problem. From Figure 13.11, we see that there are \( 4 \times 3 \times 2 \times 1 = 24 \) permutations of these four objects. We can write this number more succinctly as \( P(4, 4) = 24 \).

\[
1st \text{ letter: } 4 \times 2nd \text{ letter: } 3 \times 3rd \text{ letter: } 2 \times 4th \text{ letter: } 1
\]

\[4 \times 3 \times 2 \times 1 = 24\]

**Quiz Yourself**

a) Explain in your own words the meaning of \( P(8, 3) \).

b) Find \( P(8, 3) \).
**Some Good Advice**

It is important to remember that an equation in mathematics is a symbolic form of an English sentence. In Example 1, when we write \( P(7, 3) = 210 \), we read this as “The number of permutations of seven objects taken three at a time is 210.”

### Factorial Notation

Suppose that we want to arrange 100 objects in a straight line. The number of ways to do this is \( P(100, 100) = 100 \cdot 99 \cdot 98 \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \). Because such a product is very tedious to write out, we will introduce a notation to write such products more concisely.

**Definition**

If \( n \) is a counting number, the symbol \( n! \), called \( n \) factorial, stands for the product \( n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \cdot \ldots \cdot 2 \cdot 1 \). We define \( 0! = 1 \).

**Example 2 Using Factorial Notation**

Compute each of the following:

- a) \( 6! \)
- b) \( (8 - 3)! \)
- c) \( \frac{9!}{5!} \)
- d) \( \frac{8!}{5!3!} \)

**Solution:**

- a) \( 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 \).
- b) Remembering the Order Principle from Section 1.1, we work in parentheses first and do the subtraction before computing the factorial. Therefore, \( (8 - 3)! = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \).
- c) \( \frac{9!}{5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 9 \cdot 8 \cdot 7 \cdot 6 = 3,024 \).
  - Notice how canceling \( 5! \) from the numerator and denominator makes our computations simpler.
- d) \( \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 = 56 \).
  - Notice how canceling \( 5! \) from the numerator and denominator makes our computations simpler.

Now try Exercises 5 to 12.

In Example 1(b), we first wrote \( P(7, 3) \) in the form \( 7 \times 6 \times 5 \). This product started as though we were going to compute \( 7! \); however, the product \( 4 \cdot 3 \cdot 2 \cdot 1 \) is missing. Another way of saying this is that \( P(7, 3) = \frac{7!}{4!} = \frac{7!}{(7 - 3)!} \). We state this as a general rule.

**Formula for Computing \( P(n, r) \)**

\[
P(n, r) = \frac{n!}{(n - r)!}
\]

We will use this formula in the computations in Example 3.

**Example 3 Counting Ways to Fill Positions at a Community Theater**

The 12-person community theater group produces a yearly musical. Club members select one person from the group to direct the play, a second to supervise the music, and a third to
handle publicity, tickets, and other administrative details. In how many ways can the group fill these positions?

**Solution:** This can be viewed as a permutation problem if we consider that we are selecting 3 people from 12 and then arranging those names in a straight line—director, music, administration. The answer to this question then is

\[ P(12, 3) = \frac{12!}{(12 - 3)!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1,320 \]

ways to select 3 people from the 12 available for the three positions.

Now try Exercises 17 to 20. *

---

**Some Good Advice**

When talking with my students I often kid them by saying that to be a good mathematics student you have to be lazy—but in the right way. I don’t mean that it is good to sleep through your math class or not do homework, but it is important to try to make your computations as easy as possible for yourself. For example, in Example 3 you had to compute the expression \( P(12, 3) \). When doing this by hand, the weak student may compute this as

\[ P(12, 3) = \frac{12!}{(12 - 3)!} = \frac{479,901,600}{362,880} = 1,320. \]

Working with such large numbers is tedious, time-consuming, and prone to error. However, if you simplify the computations as we did in Example 3, the computations go much more quickly. Some may argue that if you are using a calculator, then it doesn’t make a difference as to how you do the computations. This is true provided that the numbers you are working with do not cause an overflow on your calculator. For example, try finding \( \frac{500!}{497!} \) on your calculator without canceling before dividing.

---

**Key Point**

In forming combinations, order is not important.

<table>
<thead>
<tr>
<th>Directing</th>
<th>Music</th>
<th>Administration</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>A</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>C</td>
<td>B</td>
<td>A</td>
</tr>
</tbody>
</table>

**Table 13.1** Six different assignments of responsibilities for the musical are now handled by one committee.

---

**Combinations**

In order to introduce a new idea, let’s change the conditions in Example three slightly. Suppose that instead of selecting three people to carry out three different responsibilities, we choose a three-person committee that will work jointly to see that the job gets done. In Example 3, if A, B, and C were selected to be in charge of directing, music, and administration, respectively, that would be different than if B directed, C supervised the music, and A handled administrative details. However, with the new plan, it makes no difference whether we say A, B, C or B, C, A. From Table 13.1, we see that all of the six different assignments of A, B, and C under the old plan are equivalent to a single committee under the new plan.

What we are saying for A, B, and C applies to any three people in the theater group. Thus, the answer 1,320 we found in Example 3 is too large, so we must reduce it by a factor of 6 under the new plan. Therefore, the number of three-person committees we could form would be \( \frac{1,320}{6} = 220 \). Remember that the factor of 6 we divided out is really 3! (the number of ways we can arrange three people in a straight line). This means that the number of three-person committees we can select can be written in the form

\[ \frac{1,320}{6} = \frac{P(12, 3)}{3!} \]

The reason this number is smaller is that now we are concerned only with choosing a set of people to produce the play, but the order of the people chosen is not important.
We can generalize what we just saw. If we are choosing \( r \) objects from a set of \( n \) objects and are not interested in the order of the objects, then to count the number of choices we must divide \( P(n, r) \) by \( r! \). We now state this formally.

**FORMULA FOR COMPUTING \( C(n, r) \)**

If we choose \( r \) objects from a set of \( n \) objects, we say that we are forming a **combination** of \( n \) objects taken \( r \) at a time.

The notation \( C(n, r) \) denotes the number of such combinations.\(^\dagger\)

\[
C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r! \cdot (n-r)!}.
\]

\(^\dagger\) Also, \( nC_r \) is another common notation for \( C(n, r) \).

**Some Good Advice**

In working with permutations and combinations, we are choosing \( r \) different objects from a set of \( n \) objects. The big difference is whether the order of the objects is important. If the order of the objects matters, we are dealing with a permutation. If the order does not matter, then we are working with a combination.

Do not try to use the theory of permutations or combinations when that theory is not relevant. If a problem involves something other than simply choosing different objects (and maybe ordering them), then perhaps the fundamental counting principle may be more appropriate.

**EXAMPLE 4 Using the Combination Formula to Count Committees**

a) How many three-element sets can be chosen from a set of five objects?

b) How many four-person committees can be formed from a set of 10 people?

**SOLUTION:**

a) Because order is not important in considering the elements of a set, it is clear that this is a combination problem rather than a permutation problem. The number of ways to choose three elements from a set of five is

\[
C(5, 3) = \frac{5!}{3! \cdot (5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = \frac{20}{2} = 10.
\]

b) The number of different ways to choose four elements from a set of 10 is

\[
C(10, 4) = \frac{10!}{4! \cdot (10-4)!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{210}{24} = 210.
\]

Now try Exercises 21 to 24.
We can now address the question about the lottery syndicate that we posed at the beginning of this chapter.

**EXAMPLE 5  How Many Tickets Are Necessary to Cover All Possibilities in a Lottery?**

Recall that a syndicate intended to raise $15 million to buy all the tickets for certain lotteries. The plan was that if the prize was larger than the amount spent on the tickets, then the syndicate would be guaranteed a profit. To play the Virginia lottery, the player buys a ticket for $1 containing a combination of six numbers from 1 to 44. Assuming that the syndicate has raised the $15 million, does it have enough money to buy enough tickets to be guaranteed a winner?

**SOLUTION:** Because we are choosing a combination of 6 numbers from the 44 possible, the number of different tickets possible is $C(44, 6) = 7,059,052$. Thus, the syndicate has more than enough money to buy enough tickets to be guaranteed a winner.

In addition to wondering whether $15 million would cover all combinations in a lottery, there were some other practical concerns that caused me not to invest in the syndicate.

1) Suppose the syndicate did not raise all $15 million. What would be done with the money then? Would they play lotteries in which not all tickets can be covered? Doing this could lose all the money we invested.

2) What if there are multiple winners? In this case, the jackpot is shared and it is possible that the money won will not cover the cost of the tickets. This case would result in a loss to the syndicate.

3) Is it physically possible to actually purchase 7,059,052 tickets? Because there are $60 \times 60 \times 24 \times 7 = 604,800$ seconds in a week, it would take almost 12 weeks for one person buying one ticket every second to purchase enough tickets to cover all the combinations. In Virginia, the syndicate bought only about 5 million tickets, thus opening up the possibility that all the money could have been lost!

4) How much of the money raised will be used by the syndicate for its own administrative expenses?

5) Because the jackpot is to be shared with thousands of members of the syndicate and the prize money will be paid over a 20-year period, is the amount of return better than if the money had been placed in a high-interest-bearing account?

In light of these concerns, was I too cautious? What do you think? As a final note on this matter, since the Virginia lottery episode, some states have made it illegal to play a lottery by covering all ticket combinations.

As Example 6 shows, applications of counting appear in many gambling situations.*

**EXAMPLE 6  Counting Card Hands**

a) In the game of poker, five cards are drawn from a standard 52-card deck.† How many different poker hands are possible?

b) In the game of bridge, a hand consists of 13 cards drawn from a standard 52-card deck. How many different bridge hands are there?

---

*We cover counting as it applies to gambling in Section 13.4 at the end of this chapter.

†Figure 13.1 shows that a standard 52-card deck contains 13 hearts, 13 spades, 13 clubs, and 13 diamonds. Each suit consists of A, K, Q, J, 10, 9, . . . , 3, 2. See Table 13.2, page 652, for the possible poker hands.
Chapter 13 • Counting

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Combining Counting Methods

In some situations, it is necessary to combine several counting methods to arrive at an answer. We see this technique in the next example.

Example 7  Combining Counting Methods

The division of student services at your school is selecting two men and two women to attend a leadership conference in Honolulu, Hawaii. If 10 men and nine women are qualified for the conference, in how many different ways can management make its decision?

Solution: It is tempting to say that because we are selecting four people from a possible 19, the answer is \( C(19, 4) \). But this is wrong. If we think about it, certain choices are unacceptable. For example, we could not choose all men or three women and one man.

We can use the fundamental counting principle once we recognize that the group can be formed in two stages.

Stage 1: Select the two women from the nine available. We can do this in \( C(9, 2) = \frac{9!}{2!7!} = 36 \) ways.

Stage 2: Select two men from the 10 available. We can make this choice in \( C(10, 2) = \frac{10!}{2!8!} = 45 \) ways.

Thus, choosing the women and then choosing the men can be done in \( 36 \cdot 45 = 1,620 \) ways.

Example 8 shows another situation in which you have to use several counting techniques at the same time.

Highlight

Counting and Technology

We rarely do the kind of computations that you saw in Example 6 with pencil and paper. Not only are these computations tedious but, more important, it is easy to make errors when doing pencil-and-paper calculations.

Provided that the numbers do not get too large, you can calculate the number of permutations and combinations on a graphing calculator such as the TI-83 or TI-84. To find \( C(52, 5) \) you would

1st: Type 52 (but do not press ENTER).
2nd: Press the MATH key.
3rd: Move the cursor to the right to PRB and then down to nCr (see Screen 1).
4th: Press ENTER.
5th: Type 5 and press ENTER (see Screen 2).

Notice that we get the same answer as we found in Example 6 by hand.

Solution:

a) \( C(52, 5) = \frac{52!}{5!47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960 \).

b) \( C(52, 13) = \frac{52!}{13!39!} = 635,013,559,600 \).

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Example 8 shows another situation in which you have to use several counting techniques at the same time.
EXAMPLE 8  Forming a Governing Committee

Assume that you and 15 of your friends have formed a company called Net-Media, an Internet music and video provider. A committee consisting of a president, a vice president, and a three-member executive board will govern the company. In how many different ways can this committee be formed?

SOLUTION:  We can form the committee in two stages:

a) Choose the president and vice president.

b) Select the remaining three executive members.

Thus, we can apply the fundamental counting principle.

Because order is important in stage a), we recognize that we can choose the president and vice president in \( P(16, 2) \) ways.

Order is not important in choosing the remaining three committee members, so we can select the rest of the committee from the 14 remaining people in \( C(14, 3) \) ways.

Thus, you see that we can do stage a) followed by stage b) in

\[
P(16, 2) \times C(14, 3) = 87,360 \text{ ways.}
\]

It is surprising how the same mathematical idea appears again and again in different cultures over a period of centuries. For example, Pascal’s triangle in Figure 13.13 occurs in the writings of fourteenth-century Chinese mathematicians, which some historians believe are based on twelfth-century manuscripts. Pascal’s triangle is also found in the work of eighteenth-century Japanese mathematicians.

In Pascal’s triangle, we begin numbering the rows of the triangle with zero and also begin numbering the entries of each row with zero. For example, in Figure 13.13, the number 6 is the second entry in the fourth row. Each entry in this triangle after row 1 is the sum of the two numbers immediately above it, which are a little to the right and a little to the left.

Pascal’s triangle has an interesting application in set theory. Suppose that we want to find all subsets of a particular set. Recall that the Be Systematic strategy from Section 1.1 encourages us to list possibilities systematically to solve problems. We can find all subsets of \( \{1, 2, 3, 4\} \) by first listing sets of size 0, then sets of size 1, 2, 3, and 4, as in the following list:

\[
\emptyset \quad \text{1 zero-element set}
\]

\[\begin{align*}
\{1\}, \{2\}, \{3\}, \{4\} & \quad \text{4 one-element sets} \\
\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\} & \quad \text{6 two-element sets} \\
\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} & \quad \text{4 three-element sets} \\
\{1, 2, 3, 4\} & \quad \text{1 four-element set}
\end{align*}\]

We see that there is one subset of size 0, four of size 1, six of size 2, four of size 3, and one of size 4. Note that this pattern,

\[1 \quad 4 \quad 6 \quad 4 \quad 1,\]

occurs as the fourth row in Pascal’s triangle.

Similarly, the pattern

\[1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1,\]

which is the fifth row of Pascal’s triangle, counts the subsets of size 0, 1, 2, and so on of a five-element set. In fact, the following result is true.

**Pascal’s Triangle Counts the Subsets of a Set**  The \( n \)th row of Pascal’s triangle counts the subsets of various sizes of an \( n \)-element set.

Quiz Yourself

a) What is the seventh row of Pascal’s triangle?

b) How many subsets of size 3 are there in a seven-element set?
Let us look at the subsets of the set \( S = \{1, 2, 3, 4\} \) again. The sets \( \{1\} \), \( \{2\} \), \( \{3\} \), and \( \{4\} \) are the four different ways we can choose a one-element set from \( S \). The sets \( \{1, 2\} \), \( \{1, 3\} \), \( \{1, 4\} \), \( \{2, 3\} \), \( \{2, 4\} \), and \( \{3, 4\} \) are all the different ways that we can choose a two-element set from \( S \). Because combinations are sets, this means that we can also use the entries in Pascal’s triangle to count combinations.

**EXAMPLE 9  Relating Entries in Pascal’s Triangle to Combinations**

Interpret the numbers in the fourth row of Pascal’s triangle as counting combinations.

**SOLUTION:** The fourth row of Pascal’s triangle is

\[
\begin{array}{ccccccc}
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

Because the leftmost 1 is the zeroth entry in the fourth row, we can write it as \( C(4, 0) \). That is to say, \( C(4, 0) = 1 \). The first entry in the fourth row is 4, which means that \( C(4, 1) = 4 \). Similarly, \( C(4, 2) = 6 \), \( C(4, 3) = 4 \), and \( C(4, 4) = 1 \).

For counting problems that are not too large, it is quicker to use entries in Pascal’s triangle to count combinations rather than to use the formula for \( C(n, r) \) that we stated earlier, as you will see in Example 10.

**EXAMPLE 10  Using Pascal’s Triangle to Count Drug Combinations**

Frequently, doctors experiment with combinations of new drugs to combat hard-to-treat illnesses such as AIDS and hepatitis. Assume that a pharmaceutical company has developed five antibiotics and four immune system stimulators. In how many ways can we choose a treatment program consisting of three antibiotics and two immune system stimulators to treat a disease? Use Pascal’s triangle to speed your computations.

**SOLUTION:** We can select the drugs in two stages: first select the antibiotics and then the immune system stimulators. Therefore, we can apply the fundamental counting principle, introduced in Section 13.2.

In selecting the antibiotics we are choosing three drugs from five, so this can be done in \( C(5, 3) \) ways. Looking at the fifth row of Pascal’s triangle, which is

\[
\begin{array}{ccccccc}
1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

we see that the third entry (remember, we start counting row entries with 0) is 10. Thus, \( C(5, 3) = 10 \).

Choosing two immune system stimulators from four can be done in \( C(4, 2) \) ways. The fourth row in Pascal’s triangle is

\[
\begin{array}{ccccccc}
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

This means that \( C(4, 2) = 6 \). It follows that we can choose the antibiotics and then the immune system stimulators in \( C(5, 3) \times C(4, 2) = 10 \times 6 = 60 \) ways. The accompanying screen shows how to do the calculations with a graphing calculator.
Blaise Pascal

While trying to answer questions about rolling dice and drawing cards, a group of seventeenth-century European mathematicians began to develop a theory of counting. The Frenchman Blaise Pascal, who was one of the central figures in this endeavor, wrote a paper on combinations in 1654.

Pascal was a child prodigy who became interested in Euclid’s Elements at age 12. Within four years, he wrote a research paper of such quality that some of the leading mathematicians of the time refused to believe that a 16-year-old boy had written it.

Pascal temporarily abandoned mathematics to devote himself to philosophy and religion. However, while suffering from a toothache, he decided to take his mind off the pain by thinking about geometry. Surprisingly, the pain stopped. Pascal took this as a sign from heaven that he should return to mathematics. He resumed his research but soon became seriously ill with dyspepsia, a digestive disorder. Pascal spent the remaining years of his life in excruciating pain, doing little work until his death at age 39 in 1662.

Looking Back*

These exercises follow the general outline of the topics presented in this section and will give you a good overview of the material that you have just studied.

1. In Example 1, you saw the patterns $4 \times 3 \times 2 \times 1$ and $7 \times 6 \times 5$. Why did we decrease the numbers in these products one at a time?
2. What is a quick way to compute the expression $\frac{9!}{7!}$ as we did in Example 2?
3. What point were we making in Table 13.1?
4. Why are we discussing Blaise Pascal in this section?

Sharpening Your Skills

In Exercises 5–16, calculate each value.

5. $4!$
6. $3!$
7. $(8 - 5)!$
8. $(10 - 5)!$
9. $\frac{10!}{7!}$
10. $\frac{11!}{9!}$
11. $\frac{10!}{7!3!}$
12. $\frac{11!}{9!2!}$
13. $P(6, 2)$
14. $P(5, 3)$
15. $C(10, 3)$
16. $C(10, 4)$

In Exercises 17–26, find the number of permutations.

17. Eight objects taken three at a time
18. Seven objects taken five at a time
19. Ten objects taken eight at a time
20. Nine objects taken six at a time

In Exercises 21–24, find the number of combinations.

21. Eight objects taken three at a time
22. Seven objects taken five at a time

23. Ten objects taken eight at a time
24. Nine objects taken six at a time

Explain the meaning of each symbol in Exercises 25 and 26.

25. $P(10, 3)$
26. $C(6, 2)$

27. Find the eighth row in Pascal’s triangle.
28. Find the tenth row in Pascal’s triangle.

Use the seventh row of Pascal’s triangle to answer Exercises 29 and 30.

29. How many two-element subsets can we choose from a seven-element set?
30. How many four-element subsets can we choose from a seven-element set?

In Exercises 31–34, describe where each number would be found in Pascal’s triangle.

31. $C(18, 2)$
32. $C(19, 5)$
33. $C(20, 6)$
34. $C(21, 0)$

Applying What You’ve Learned

In Exercises 35–46, specify the number of ways to perform the task described. Give your answers using $P(n, r)$ or $C(n, r)$ notation. The key in recognizing whether a problem involves permutations or combinations is deciding whether order is important.

35. Quiz possibilities. On a biology quiz, a student must match eight terms with their definitions. Assume that the same term cannot be used twice.
36. Scheduling interviews. Producers wish to interview six actors on six different days to replace the actor who plays the phantom in Broadway’s longest running play, The Phantom of the Opera.
37. Choosing tasks. Gil Grissom on CSI wants to review the files on three unsolved cases from a list of 17. He does not care about the order in which he reads the files.

*Before doing these exercises, you may find it useful to review the note How to Succeed at Mathematics on page xix.
38. **Selecting roommates.** Annette has rented a summer house for next semester. She wants to select four roommates from a group of six friends.

39. **Baseball lineups.** Joe Torre has already chosen the first, second, fourth, and ninth batters in the batting lineup. He has selected five others to play and wants to write their names in the batting lineup.

40. **Race results.** There are seven boats that will finish the America’s Cup yacht race.

41. **Journalism awards.** Ten magazines are competing for three identical awards for excellence in journalism. No magazine can receive more than one award.

42. **Journalism awards.** Redo Exercise 41, but now assume the awards are all different. No magazine can receive more than one award.

43. **Lock patterns.** In order to unlock a door, five different buttons must be pressed down on the panel shown. When the door opens, the depressed buttons pop back up. The order in which the buttons are pressed is not important.

44. **Lock patterns.** A bicycle lock has three rings with the letters A through K on each ring. To unlock the lock, a letter must be selected on each ring. Duplicate letters are not allowed, and the order in which the letters are selected on the rings does not matter.

45. **Choosing articles for a magazine.** The editorial staff of Oprah’s *O Magazine* is reviewing 17 articles. They wish to select eight for the next issue of the magazine.

46. **Choosing articles for a magazine.** Redo Exercise 45, except that in addition to choosing the stories, the editorial staff must also decide in what order the stories will appear.

47. **Reading program.** Six players are to be selected from a 25-player Major League Baseball team to visit schools to support a summer reading program. In how many ways can this selection be made?

48. **Reading program.** If in Exercise 47 we not only want to select the players but also want to assign each player to visit one of six schools, how many ways can that assignment be done?

49. **Writing articles.** In Exercise 46, if it takes 1 minute to write a list of eight articles selected for the magazine, how many years would it take to write all possible lists of 8 articles? Assume 60 minutes in an hour, 24 hours in a day, and 365 days in a year.

50. **Writing assignments.** In Exercise 48, if it takes 1 minute to write a list of six players with their assignments, how many years would it take to write all possible lists of assignments?

A typical bingo card is shown in the figure. The numbers 1–15 are found under the letter B, 16–30 under the letter I, 31–45 under the letter N, 46–60 under the letter G, and 61–75 under the letter O. The center space on the card is labeled “FREE.”

51. **Bingo cards.** How many different columns are possible under the letter B?

52. **Bingo cards.** How many different columns are possible under the letter N?

53. **Bingo cards.** How many different bingo cards are possible? (*Hint: Use the fundamental counting principle.*)

54. **Bingo cards.** Why is the answer to Exercise 53 not $P(75, 24)$?

55. **Selecting astronauts.** NASA wants to appoint two men and three women to send back to the moon in 2018. The finalists for these positions consist of six men and eight women. In how many ways can NASA make this selection?

56. **Computer passwords.** A password for a computer consists of three different letters of the alphabet followed by four different digits from 0 to 9. How many different passwords are possible?

57. **Forming a public safety committee.** Nicetown is forming a committee to investigate ways to improve public safety in the town. The committee will consist of three representatives from the seven-member town council, two members of a five-person citizens advisory board, and three of the 11 police officers on the force. How many ways can that committee be formed?

58. **Choosing a team for a seminar.** HazMat, Inc., is sending a group of eight people to attend a seminar on toxic waste disposal. They will send two of eight engineers and three of nine crew supervisors and the remainder will be chosen from the five senior managers. In how many ways can these choices be made?

59. **Designing a fitness program.** To lose weight and shape up, Sgt. Walden is considering doing two types of exercises to
improve his cardiovascular fitness from running, bicycling, swimming, stair stepping, and Tae Bo. He is also going to take two nutritional supplements from AllFit, Energize, ProTime, and DynaBlend. In how many ways can he choose his cardiovascular exercises and nutritional supplements?

60. Counting assignment possibilities. Gina must write evaluation reports on three hospitals and two health clinics as part of her degree program in community health services. If there are six hospitals and five clinics in her vicinity, in how many ways can she complete her assignment?

61. Picking a team for a competition. The students in the 12-member advanced communications design class at Center City Community College are submitting a project to a national competition. They must select a four-member team to attend the competition. The team must have a team leader and a main presenter; the other two members have no particularly defined roles. In how many different ways can this team be formed?

62. Why is Exercise 61 neither a strictly permutations nor a strictly combinations problem?

63. Choosing an evaluation team. The academic computing committee at Sweet Valley College is in the process of evaluating different computer systems. The committee consists of five administrators, seven faculty, and four students. A five-person subcommittee is to be formed. The chair and vice chair of the committee must be administrators; the remainder of the committee will consist of faculty and students. In how many ways can this subcommittee be formed?

64. Why is Exercise 63 neither a strictly permutations nor a strictly combinations problem?

65. Armored car routes. How many different direct routes can the driver take from the shopping mall to the bank?

66. Armoured car routes. How many direct routes can the driver take from the mall to the bank if he must also stop at the jewelry store?

67. Communicating Mathematics

67. How do you distinguish a combination problem from a permutation problem?

68. In Example 7, we were choosing 4 people from a group of 19, yet our answer was not \( C(19, 4) \). Explain this. How does the fundamental counting principle apply here?

69. Where would you find the number \( C(n, r) \) in Pascal’s triangle?

70. Recall that combinations are subsets of a given set and that \( C(n, r) \) is the number of subsets of size \( r \) in an \( n \)-element set. Give an intuitive explanation as to why you would expect each of the following equations to be true.

a. \( C(5, 0) = 1 \)

b. \( C(8, 7) = 8 \)

71. Consider the question: How many four-digit numbers can be formed that are odd and greater than 5,000? Why is this not a permutation problem?

72. Consider the question: How many four-digit numbers can be formed using the digits 1, 3, 5, 7, and 9? Repetition of digits is not allowed. Why is this not a combination problem?

73. Whenever we used the formulas for computing \( P(n, r) \) or \( C(n, r) \), we saw that many factors from the numerator canceled with all the factors from the denominator so that the resulting number was never a fraction. Using the definitions of \( P(n, r) \) and \( C(n, r) \), explain why these numbers can never be fractions.

74. Explain why \( C(n, r) = C(n, n - r) \). To understand this statement, recall the Three-Way Principle from Section 1.1, which states that making numerical examples is useful in gaining understanding of a situation.

75. We numbered the rows of Pascal’s triangle beginning with 0 instead of 1. Explain why we did this.

76. We numbered the entries of each row in Pascal’s triangle beginning with 0 instead of 1. Explain why we did this.

Using Technology to Investigate Mathematics

77. Learn to use either a graphing calculator or an Excel spreadsheet to find \( P(n, r) \) and \( C(n, r) \), and then solve some of the exercises in this section using those technologies.

78. Search the Internet for applets that compute \( P(n, r) \) and \( C(n, r) \) and use them to solve some of the exercises in this section.

For Extra Credit

79. Playing cards. How many five-card hands chosen from a standard deck contain two diamonds and three hearts?

80. Playing cards. How many five-card hands chosen from a standard deck contain two kings and three aces?

The following diagram shows a portion of Pascal’s triangle found in a book called The Precious Mirror, written in China in 1303. Use this diagram to answer Exercises 81–84.
81. What symbol do you expect to find in the circle labeled 81?
82. What symbol do you expect to find in the circle labeled 82?
83. What number does the symbol \( \binom{n}{r} \) represent?
84. What number does the symbol \( \binom{n-1}{r} \) represent?

Consider rows \( n \) and \( n - 1 \) of Pascal’s triangle, shown in the diagram to the right. Use this diagram to answer Exercises 85–89.

85. What expression will be in the circle labeled 85?
86. What expression will be in the circle labeled 86?
87. Complete the following equation: \( \square + \binom{n-1}{r} = \square \)
88. Complete the following equation: \( \binom{n}{r} + \square = \square \)
89. Notice that the arrangement of numbers in each row of Pascal’s triangle is symmetric. See how the 4s are balanced in the fourth row, the 5s are balanced in the fifth row, and so on. Can you give a set theory explanation to account for this?