Hyperplane arrangements
Lecture 26

November 21, 2017

Cross-cut theorem
Definitions
Region recurrence
Characteristic polynomial recurrence
Zaslavsky’s Theorem
Usual end
Last time

1. Möbius functions

\[ \mu(x, y) = \begin{cases} 
1 & \text{if } x = y \\
- \sum_{z: x < z \leq y} \mu(z, y) & \text{otherwise}
\end{cases} \]

2. Rank selection- generalized \( \alpha \) and \( \beta \)
Cross-cut

Let $L$ be a lattice. An upper crosscut of $L$ is a set $x \subseteq L - \{\hat{1}\}$ such that if $y \in (L - X) - \{\hat{1}\}$, then $y < x$ for some $x \in X$. A lower crosscut of $L$ is a set $x \subseteq L - \{\hat{0}\}$ such that if $y \in (L - X) - \{\hat{0}\}$, then $y > x$ for some $x \in X$. 
Rota’s Crosscut Theorem

Theorem (Wiesner’s theorem)

Let $L$ be a finite lattice with at least 2 elements and let $a \in L$ such that $a \neq \hat{1}$. Then

$$\sum_{x : x \wedge a = \hat{0}} \mu(x, \hat{1}) = 0.$$ 

Theorem (Rota’s Crosscut Theorem)

Let $L$ be a finite lattice and let $X$ be an upper crosscut. Then

$$\mu(\hat{0}, \hat{1}) = \sum_{Y \subseteq X : \wedge Y = \hat{0}} (-1)^{|Y|}.$$ 

Similarly, if $X$ is a lower crosscut, then

$$\mu(\hat{0}, \hat{1}) = \sum_{Y \subseteq X : \vee Y = \hat{1}} (-1)^{|Y|}.$$
Hyperplane arrangements

$V$ is an $n$ dimensional vector space over a field $\mathbb{K} = \mathbb{R}$.

A linear hyperplane $H$ is an $(n - 1)$-dimensional subspaces of $V$:

$$H = \{v \in V : v \cdot n_H = 0\},$$

where $n_H \neq 0$, $n_H \in V$.

A affine hyperplane $H$ is an $(n - 1)$-dimensional subspaces of $V$:

$$H = \{v \in V : v \cdot n_H = a\},$$

where $n_H \neq 0$, $n_H \in V$, $a \in \mathbb{R}$.

A hyperplane arrangement is a finite set $\mathcal{A}$ of (possibly) affine hyperplanes. I will usually leave off the affine.
Dimension and rank

The **dimension** $\dim \mathcal{A}$ is $\dim V$.

The **rank** $\text{rank}(\mathcal{A})$ is the dimension of the subspace $B(\mathcal{A})$ spanned by the normals to hyperplanes in $\mathcal{A}$.

$$\text{ess}(\mathcal{A}) = \{ H \cap B(\mathcal{A}) : H \in \mathcal{A} \} \subseteq B(\mathcal{A}).$$
The braid arrangement $\text{Br}_n$ is the set of hyperplanes $x_i = x_j$ in $\mathbb{R}^n$. 
The intersection poset $L(A)$ of an arrangement $A$ is the poset of all nonempty intersections of subsets of $A$, ordered by reverse inclusion. By convention, $\hat{0}$ is $V$. An arrangement $A$ is called central if $\bigcap_{H \in A} H \neq \emptyset$. 
Examples
Meet-semilattice

The poset $L(A)$ is a meet-semilattice.
Meet given by

$$\left( \bigcap_{H \in B} H \right) \land \left( \bigcap_{H \in C} H \right) = \bigcap_{H \in B \cap C} H,$$

for $B, C \subseteq A$. 

Definitions 10/23
Polynomials

The characteristic polynomial of $\mathcal{A}$ is

$$\chi_{\mathcal{A}}(x) = \sum_{t \in \mathcal{A}} \mu(\hat{0}, t)x^{\dim(t)}.$$ 

Every hyperplane is the zero set of some linear form and so $\mathcal{A}$ is the zero set of their product. We can specify an arrangement by that product, called the defining polynomial.
The regions of $A$ are the connected components of $V - A$. A region is relatively bounded if the corresponding region in $\text{ess}(A)$ is bounded. Let $r(A)$ be the number of regions of $A$ and let $b(A)$ be the number of relatively bounded regions of $A$. 

The braid arrangement $B_{n}$ is the set of hyperplanes $x_i = x_j$ in $\mathbb{R}^n$. 
Some subarrangements

Let $x \in L(\mathcal{A})$ and define

$$A_x = \{H \in \mathcal{A} : x \subseteq H\}$$

and

$$A^x = \{H \cap x : H \in \mathcal{A} - A_x\}.$$
Region recurrence

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$ and let $H \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} - \{H\}$ and let $\mathcal{A}'' = \mathcal{A}^H$. Then

1. $r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'')$

2. $b(\mathcal{A}) = \begin{cases} b(\mathcal{A}') + b(\mathcal{A}'') & \text{if rank}(\mathcal{A}) = \text{rank}(\mathcal{A}') \\ 0 & \text{if rank}(\mathcal{A}) = \text{rank}(\mathcal{A}') + 1 \end{cases}$
Whitney’s formula

For any real hyperplane arrangement \( \mathcal{A} \), we have

\[
\chi_{\mathcal{A}}(x) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} x^{n - \text{rank}(\mathcal{B})}.
\]

\( \chi_{\mathcal{A}}(x) \) is central.
Example

Characteristic polynomial recurrence
Let $H \in \mathcal{A}$. Let $\mathcal{A'} = \mathcal{A} - \{H\}$ and let $\mathcal{A''} = \mathcal{A}^H$. Then

$$
\chi_{\mathcal{A}}(x) = \chi_{\mathcal{A'}}(x) - \chi_{\mathcal{A''}}(x).
$$
Zaslavsky’s Theorem

Let $\mathcal{A}$ be an essential arrangement, $\dim(V) = n$. Then

$$t(\mathcal{A}) = (-1)^n \chi_\mathcal{A}(-1) \text{ and } t(\mathcal{A}) = (-1)^n \chi_\mathcal{A}(1).$$
Next time(s)

1. Eulerian posets
2. cd-index
End of lecture 26

1.