Modular and geometric lattices
Lecture 18

October 24, 2017
Last time

Distributive and modular lattices.
1. (JD) Draw the subspace lattice when $q = n = 2$.
2. (Jennifer) $D_n$ is distributive.
3. Modular and semimodular lattices
4. geometric
Next time

Volunteers for

1. (Ahlam) FTFDL set up. Show that $\phi : L \rightarrow J(\text{Irr}(L))$ given by

   \[ \phi(x) = \langle p | p \in \text{Irr}(L), p \leq_L x \rangle \]

   is a lattice isomorphism.

2. (JD) FTFDL set up. Show that an order ideal in $P$ is join-irreducible in $J(P)$ if and only if it is principal.

3. (Stephanie) Show the two conditions for the distributive law are equivalent.
Leftovers

Is a lattice distributive if and only if its dual is distributive?
Up to isomorphism, the finite distributive lattices are exactly the lattices $J(P)$, where $P$ is a finite poset. Moreover, $L \cong J(\text{Irr}(L))$ for every lattice $L$ and $P \cong \text{Irr}(J(P))$ for every poset $P$. 
A lattice $L$ is **modular** if for every $x, y, z \in L$ with $x \leq z$ satisfy

\[ x \lor (y \land z) = (x \lor y) \land z. \]
Characterizations of modularity

Let $L$ be a lattice. Then the following are equivalent.

1. $L$ is modular.
2. For all $x, y, z \in L$, if $x \in [y \land z, z]$ then $x = (x \lor y) \land z$.
3. For all $x, y, z \in L$, if $x \in [y, y \lor z]$ then $x = (x \land z) \lor y$.
4. For all $y, z \in L$, there is an isomorphism of lattices
   
   $$[y \land z, z] \cong [y, y \lor z].$$
$N_5$ and $M_5$
Obstructions

Theorem

Let $L$ be a lattice.

1. $L$ is modular if and only if it contains no sublattice isomorphic to $N_5$.

2. $L$ is distributive if and only if it contains no sublattice isomorphic to $N_5$ or $M_5$. 
Finite upper semi-modular lattices—review from a few weeks ago

**Proposition**

Let $L$ be a finite lattice. The following two conditions are equivalent.

1. $L$ is graded and the rank function $\rho$ of $L$ satisfies
   
   $$\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y) \text{ for all } x, y \in L.$$

2. If $x$ and $y$ cover $x \wedge y$, then $x \vee y$ covers both $x$ and $y$.

**Definition**

A lattice $L$ is upper semimodular if for all $x, y \in L$,

$$x \wedge y \leq y \Rightarrow x \leq x \vee y.$$  

The lattice $L$ is lower semimodular if the converse holds.
Lemma

If $L$ is modular, then it is upper and lower semimodular.
Modular

Theorem
A lattice \( L \) is modular if and only if it is ranked with rank function \( \rho \) satisfying

\[
\rho(x \lor y) + \rho(x \land y) = \rho(x) + \rho(y).
\]
Definitions

1. Atom
2. Coatom
3. Atomic
4. Vector subspace
5. Affine subspace
Example

$V$ is a vector space over a field $\mathbb{K}$. Let $E$ be a finite subset of $V$.

$L(E) = \{ W \cap E | W \subseteq V \text{ is a vector subspace} \}$.

Lattice operations: $(W \cap E) \wedge (X \cap E) = (W \cap X) \cap E$ and $(W \cap E) \vee (X \cap E) = (W + X) \cap E$

Example of an example: $V = \mathbb{R}^2$, $E = \{a, b, c, d\}$, where $a = (-1, 1)$, $b = (0, 0)$, $c = (-1, 1)$, $d = (1, 1)$. 
Example

\( V \) is a vector space over a field \( \mathbb{K} \). Let \( E \) be a finite subset of \( V \).

\[ L^{\text{aff}}(E) = \{ W \cap E | W \subseteq V \text{ is an affine subspace} \}. \]

Lattice operations: \((W \cap E) \wedge (X \cap E) = (W \cap X) \cap E \) and \((W \cap E) \vee (X \cap E) = (W + X) \cap E \)

Example of an example: \( V = \mathbb{R}^2, E = \{ a, b, c, d \} \), where \( a = (-1, 1), b = (0, 0), c = (-1, 1), d = (1, 1) \).
Example

$V$ is a vector space over a field $\mathbb{K}$. Let $E$ be a finite subset of $V$.

$L^{\text{aff}}(E) = \{ W \cap E | W \subseteq V \text{ is an affine subspace} \}$.

Example of an example: $V = \mathbb{R}^2$, $E = \{a, b, c, d, e\}$, where $\{a, b, c\}$ are collinear, $\{a, d, e\}$ are collinear, and there are no other collinear triples.
More on this example

1. \( L(E) \cong \{ \mathbb{K}A | A \subset E \} \)
2. \( L(E) \) is always an atomic lattice.
Definition
A lattice $L$ is \textit{geometric} if it is (upper) semimodular and atomic. If $L \cong L(E)$ for some set of vectors $E$, we say that $E$ is a linear representation of $L$. 
Definitions

1. Complemented
2. Relatively complemented
3. Flats
Proposition 3.3.3

Proposition

Let $L$ be a finite semimodular lattice. Then the following two conditions are equivalent.

1. $L$ is relatively complemented.
2. $L$ is atomic.
Next time

More on posets.
End of lecture 18

Volunteers for

1. Suppose $L$ is semimodular and let $q, r, s \in L$. If $q \lessdot r$, then either $q \lor s = r \lor s$ or $q \lor s \lessdot r \lor s$.

2. Show that $u' \land y = x$ in the proof of Proposition 3.3.3.