Rational functions with nonnegative power series coefficients.

Ira M. Gessel

Department of Mathematics
Brandeis University

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Rational functions with nonnegative power series coefficients

Consider the following three rational functions:

\[ A(x, y, z) = \frac{1}{1 - 2(x + y + z) + 3(xy + xz + yz)} \]

\[ B(x, y, z) = \frac{1}{1 - x - y - z + 4xyz} \]

\[ C(x, y, z) = \frac{1}{1 - x - y - xz - yz + 4xyz} = \frac{1}{1 - (1 + z)(x + y)} + 4xyz \]

All three are known to have nonnegative coefficients (Szeg˝o, Kaluza 1933).
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$$A(x, y, z) = \frac{1}{(1 - x)(1 - y)(1 - z)} B \left( \frac{x}{1 - x}, \frac{y}{1 - y}, \frac{z}{1 - z} \right)$$

and

$$B(x, y, z) = \frac{1}{1 - z} C \left( x, y, \frac{z}{1 - z} \right).$$

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So it suffices to show that the coefficients of $C$ are nonnegative.

But first, there is another way to show that the coefficients of $B(x, y, z)$ are nonnegative (due to Richard Askey):
Askey’s proof

The coefficients of

\[ D(x, y, z) = \frac{\sqrt{1 - 4xy}}{1 - x - y - z + 4xyz} \]

are nonnegative.

Therefore so are the coefficients of

\[ B(x, y, z) = \frac{1}{\sqrt{1 - 4xy}} D(x, y, z). \]
$D(x, y, z)$ is nonnegative because there is an explicit formula for its coefficients:

$$D(x, y, z) = \sum_{a,b,c} S(a, b, c)x^a y^b z^c,$$

where

$$S(a, b, c) = \begin{cases} 
\frac{(a - b + c)! (a + b - c - 1)!}{(a - b - c - 1)! a! b! c!}, & \text{if } a > b + c \\
S(b, a, c), & \text{if } b > a + c \\
\frac{(a - b + c)! (b + c - a)!}{(c - a - b)! a! b! c!}, & \text{if } a + b \leq c \\
0, & \text{otherwise}
\end{cases}$$
This follows by equating coefficients in

\[(1 - x - y - z + 4xyz)D(x, y, z) = \sqrt{1 - 4xy}\]

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or from a hypergeometric identity: We have

$$\frac{\sqrt{1 - 4xy}}{1 - x - y - z + 4xyz} = \sum_{c=0}^{\infty} \frac{(1 - 4xy)^{c+\frac{1}{2}}}{(1 - x - y)^{c+1}} z^c.$$ 

Using the Pfaff-Saalschütz theorem we obtain

$$\frac{(1 - 4xy)^{\gamma + \frac{1}{2}}}{(1 - x - y)^{\gamma + 1}} = \sum_{a, b} \frac{(-\gamma)_{a+b}(\gamma + 1)_{a-b}}{a! b! (-\gamma)_{a-b}} x^a y^b;$$

we need to take a limit as $\gamma \to c$.

Here $(\gamma)_n = \gamma(\gamma + 1) \cdots (\gamma + n - 1)$, $(\gamma)_{-n} = (-1)^n/(1 - \gamma)_n$. 
The numbers $S(a, b, c)$ are “super ballot numbers”. For $c = 0$ they reduce to the ballot numbers:

$$S(a, b, 0) = \frac{a - b}{a + b} \binom{a + b}{a}, \text{ for } a > b.$$ 

Also interesting is the special case of “super Catalan numbers”:

$$T(m, n) = S(m + n, n, m - 1) = \frac{1}{2} \frac{(2m)! (2n)!}{m! n! (m + n)!}.$$
The numbers $S(a, b, c)$ are “super ballot numbers". For $c = 0$ they reduce to the ballot numbers:

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In particular,

$$T(1, n) = C_n = \frac{(2n)!}{n! (n + 1)!}$$

$$T(2, n) = 6 \frac{(2n)!}{n! (n + 2)!} = 4C_n - C_{n+1}$$

Nice combinatorial interpretations of $T(2, n)$ are known but not for $T(m, n)$ for $m > 2$. 
A positive compositional inverse

There is an interesting consequence of the positivity of

\[ A(x, y, z) = \frac{1}{1 - 2(x + y + z) + 3(xy + xz + yz)}. \]

**Theorem.** Let \( f = f(x) \) be the compositional inverse of the power series \( x - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x^3 \). Then \( f \) has positive coefficients.
Proof. We have  
\[ f - (\alpha + \beta + \gamma)f^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)f^3 = x. \]
Differentiating with respect to \( f \) and simplifying gives

\[ f' = \frac{1}{1 - 2(\alpha + \beta + \gamma)f + 3(\alpha\beta + \alpha\gamma + \beta\gamma)f^2} = A(\alpha f, \beta f, \gamma f). \]

So the positivity of \( A(x, y, z) \) implies a recurrence for the coefficients of \( f \) with positive coefficients.
In the specialization $\alpha = \beta = \gamma = 1$, $f$ reduces to

$$1 - \frac{3\sqrt{1 - 9x}}{3} = x + 3x^2 + 15x^3 + 90x^4 + 594x^5 + 4158x^6 + \cdots$$

a Catalan number generating function analogue with no known combinatorial interpretation.
Specializations

In the specialization $\alpha = \beta = \gamma = 1$, $f$ reduces to

$$\frac{1 - \sqrt[3]{1 - 9x}}{3} = x + 3x^2 + 15x^3 + 90x^4 + 594x^5 + 4158x^6 + \cdots$$

a Catalan number generating function analogue with no known combinatorial interpretation.

Also $\alpha = 1, \beta = \gamma = 0$ gives Catalan numbers and $\alpha = \beta = 1, \gamma = 0$ gives the numbers $\frac{1}{n+1} \binom{3n+1}{n}$. 
Positivity of $C(x, y, z)$

Why are the coefficients of

$$C(x, y, z) = \frac{1}{1 - x - y - xz - yz + 4xyz}$$

nonnegative?
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More generally, let

$$E(x, y, z; \lambda) = \frac{1}{1 - (1 - \lambda)x - \lambda y - \lambda xz - (1 - \lambda)yz + xyz}$$

$$= \sum_{i, j, k} \alpha(i, j, k; \lambda) x^i y^j z^k.$$ 

Note that $C(x, y, z) = E(2x, 2y, z; 1/2)$. 
Ismail and Tamhankar (1979) showed, using MacMahon’s master theorem, that

- If $i + j < k$ then $\alpha(i, j, k; \lambda) = 0$.
- If $i + j \geq k$ then

$$\alpha(i, j, k; \lambda) = \lambda^{2i+j-k}(1 - \lambda)^{k-i} \frac{(i + j - k)! k!}{i! j!}$$

$$\times \left[ \sum_m \binom{i}{m} \binom{j}{k - i + m} (1 - \lambda^{-1})^m \right]^2$$

which is clearly nonnegative for $0 < \lambda < 1$. 
A generalization of Ismail and Tamhankar’s result

Let $A = (a_{ij})$ be an $m \times n$ matrix. Let $r = (r_1, \ldots, r_n)$ and $s = (s_1, \ldots, s_m)$ be sequences of nonnegative integers, and let $r = r_1 + \cdots + r_n$ and $s = s_1 + \cdots + s_m$. We define $F_A(r, s)$ and $G_A(r, s)$ by

\[
F_A(r, s) = [y_1^{s_1} y_2^{s_2} \cdots y_m^{s_m}]
\]

\[
\left(1 + \sum_{i=1}^{m} a_{i1} y_i\right)^{r_1} \cdots \left(1 + \sum_{i=1}^{m} a_{in} y_i\right)^{r_n}
\]

and for $r \geq s$,

\[
G_A(r, s) = [x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}]
\]

\[
\left(\sum_{j=1}^{n} x_j\right)^{r-s} \left(\sum_{j=1}^{n} a_{1j} x_j\right)^{s_1} \cdots \left(\sum_{j=1}^{n} a_{mj} x_j\right)^{s_m}
\]

If $r < s$ then $G_A(r, s) = F_A(r, s) = 0$. 

It’s easy to show that

\[
\begin{pmatrix}
  r \\
  r_1, r_2, \ldots, r_n
\end{pmatrix}
\mathcal{F}_A(r, s) = 
\begin{pmatrix}
  r \\
  r - s, s_1, s_2, \ldots, s_m
\end{pmatrix}
\mathcal{G}_A(r, s).
\]
Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. Let $M$ be the $(n + m) \times (n + m)$ matrix
\[
\begin{pmatrix}
J & B^t \\
A & 0
\end{pmatrix},
\]
where $J$ is an $n \times n$ matrix of ones and $0$ is an $m \times m$ matrix of zeros, and let $Z$ be the $(n + m) \times (n + m)$ diagonal matrix with diagonal entries $x_1, \ldots, x_n, y_1, \ldots, y_m$. Then
\[
\sum_{r, s} F_A(r, s) G_B(r, s) x_1^{r_1} \cdots x_n^{r_n} y_1^{s_1} \cdots y_m^{s_m} = \frac{1}{\det(I - ZM)}.
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\sum_{r,s} F_A(r, s) G_B(r, s) x_1^{r_1} \cdots x_n^{r_n} y_1^{s_1} \cdots y_m^{s_m} = 1 / \det(I - ZM).
\]

**Proof:** Use MacMahon’s Master Theorem.
Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. Let $M$ be the $(n + m) \times (n + m)$ matrix
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**Proof:** Use MacMahon’s Master Theorem.

So if $A = B$ then each coefficient of $1 / \det(I - ZM)$ is a positive integer times the square of a polynomial in the $a_{ij}$, and if the $a_{ij}$ are real numbers then $1 / \det(I - ZM)$ has nonnegative coefficients.
**Example**

Take $A = B = [a \ b]$, and write $x$ for $x_1$, $y$ for $x_2$, and $z$ for $y_1$. Then the matrix is

$$I - \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \begin{pmatrix} 1 & 1 & a \\ 1 & 1 & b \\ a & b & 0 \end{pmatrix} = \begin{pmatrix} 1 - x & -x & -ax \\ -y & 1 - y & -by \\ -az & -bz & 1 \end{pmatrix}$$

with determinant

$$1 - x - y - a^2xz - b^2yz + (a - b)^2xyz.$$ 

Thus the coefficients of

$$\frac{1}{1 - x - y - a^2xz - b^2yz + (a - b)^2xyz}$$

are positive integers times squares of polynomials in $a$ and $b$. 
This is really the same as Ismail and Tamhankar’s result: replace $x$ with $(1 - \lambda)x$ and $y$ with $\lambda y$ and set $a = \sqrt{\frac{\lambda}{1 - \lambda}}$ and $b = -\sqrt{(1 - \lambda)/\lambda}$. 

Unfortunately, in other cases there does not seem to be an analogous nice specialization so the general theorem doesn’t seem to give generalizations of the rational functions $A(x, y, z)$ and $B(x, y, z)$. 
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Setting $a = 1, b = -1$ gives

$$C(x, y, z) = \frac{1}{1 - x - y - xz - yz + 4xyz}.$$
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A product generating function
We find also that (writing $i$ for $r_1$, $j$ for $r_2$, and $k$ for $s_1$), we have more generally

\[
\frac{1}{1 - x - y - a_1 b_1 x z - a_2 b_2 y z + (a_1 - a_2)(b_1 - b_2) x y z}
= \sum_{i,j,k} F_{i,j,k}(a_1, a_2) G_{i,j,k}(b_1, b_2) x^i y^j z^k,
\]

where

\[
F_{i,j,k}(a_1, a_2) = \sum_m \binom{i}{m} \binom{j}{k-m} a_1^m a_2^{k-m}
\]

and

\[
G_{i,j,k}(b_1, b_2) = \sum_m \binom{k}{m} \binom{i+j-k}{i-m} b_1^m b_2^{k-m}.
\]

Also

\[
F_{i,j,k}(a_1, a_2) / G_{i,j,k}(a_1, a_2) = \frac{i! j!}{k! (i + j - k)!}.
\]
The coefficients on the left side can be written as a double sum.
The coefficients on the left side can be written as a double sum. The identity we obtain by equating this double sum with the right side is equivalent to a formula of Watson:

\[
\frac{(\beta - \alpha)n}{(\beta)_n} \sum_{i,j} \frac{(-n)_{i+j}(\alpha)_{i+j}}{i! j! (\beta)_i(1-n-\beta+\alpha)_j} (xy)^i ((1-x)(1-y))^j
\]

\[
= {}_{2}F_{1} \left( \begin{array}{c} -n, \alpha \\ \beta \end{array} \Bigg| x \right) {}_{2}F_{1} \left( \begin{array}{c} -n, \alpha \\ \beta \end{array} \Bigg| y \right)
\]

(The left side is Appell’s $F_4$.)
A conjecture

We may write

$$B(x_1, x_2, x_3) = \frac{1}{1 - x_1 - x_2 - x_3 + 4x_1x_2x_3} = \frac{1}{1 - e_1 + 4e_3},$$

where $e_n$ is the $n$th elementary symmetric function in $x_1, x_2, x_3$. 
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where \( e_n \) is the \( n \)th elementary symmetric function in \( x_1, x_2, x_3 \).

In four variables, \((1 - e_1 + 4e_3)^{-1}\) does not have nonnegative coefficients, but Koornwinder (1978), and later Gillis, Reznick, and Zeilberger (1983), proved that in four variables \((1 - e_1 + 4e_3 - 16e_4)^{-1}\) has nonnegative coefficients.
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Can we extend this to more variables?
Conjecture.

\[ \frac{1}{1 - e_1 + 4e_3 - 16e_4} \]

has nonnegative coefficients in infinitely many variables.
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\[
\frac{1}{1 - e_1 + 4e_3 - 16e_4}
\]

has nonnegative coefficients in infinitely many variables.

Checked for terms up to degree 21 (using John Stembridge’s SF package for Maple).