PARKING FUNCTIONS
PARKING FUNCTIONS

and
PARKING FUNCTIONS

and

LABELLED BAR DIAGRAMS

JOINT WORK WITH GUOCÉ XIN
PARKING FUNCTIONS

and

LABELLED BAR DIAGRAMS

JOINT WORK WITH GUOCE XIN

STARTING FROM THE THESIS OF ANGELA HICKS
PARKING FUNCTIONS

and

LABELLED BAR DIAGRAMS

JOINT WORK WITH GUOCE XIN

STARTING FROM THE THESIS OF ANGELA HICKS
PARKING FUNCTIONS

and

LABELLED BAR DIAGRAMS

JOINT WORK WITH GUOCE XIN

STARTING FROM THE THESIS OF ANGELA HICKS
PARKING FUNCTIONS

and

LABELLED BAR DIAGRAMS

JOINT WORK WITH GUOCE XIN

STARTING FROM THE THESIS OF ANGELA HICKS
PARKING FUNCTIONS

and

LABELLED BAR DIAGRAMS

JOINT WORK WITH GUOCE XIN

STARTING FROM THE THESIS OF ANGELA HICKS
PARKING FUNCTIONS

and

LABELLED BAR DIAGRAMS

JOINT WORK WITH GUOCE XIN

STARTING FROM THE THESIS OF ANGELA HICKS
PARKING FUNCTIONS

and

LABELLED BAR DIAGRAMS

JOINT WORK WITH GUOCE XIN

STARTING FROM THE THESIS OF ANGELA HICKS
A Parking Function
A Parking Function
A Parking Function
A Parking Function
A Parking Function
A Parking Function

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>11</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>9</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>1</td>
<td>10</td>
<td>7</td>
<td>5</td>
</tr>
</tbody>
</table>

Saturday, January 13, 18
A Parking Function
A Parking Function
A Parking Function
A Parking Function
A Parking Function
A Parking Function
A Parking Function
A Parking Function

The diagram illustrates a parking function, which is a sequence of non-negative integers that can be represented as an increasing path in a grid. The numbers at each step in the path correspond to the sequence values. In this case, the sequence starts with 3 and includes 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and ends with 12.
A Parking Function

The diagram shows a parking function, which is a non-decreasing sequence of integers that can be visualized as a shape in the coordinate plane. The sequence is:

0 1 2 3 1 2 3 1 0 1 2 0

The shape of the parking function is represented by the shaded area in the grid.
A Parking Function
A Parking Function

0 1 2 3 1 2 3 1 0 1 2 0

3 5 7 10 1 2 8 11 4 6 9 12
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5,6 \mid 2,4,7,8 \mid 1,3 \]
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

\[
\begin{array}{c|c|c|c}
5 & 6 & 2 & 1 \\
2 & 2 & 4 & 1 \\
& 1 & 7 & 1 \\
& & 8 & 1 \\
& & & 1 \\
& & & 3 \\
\end{array}
\]
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

\[
\begin{array}{cccc}
5 & 6 & 2 & 2 \\
2 & 1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
2 & 4 & 7 & 8 \\
1 & 1 & 1 & 0 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 8 & 2 & 6 \\
0 & 1 & 1 & 2 \\
\end{array}
\quad
\begin{array}{cccc}
3 & 4 & 5 & 7 \\
0 & 1 & 2 & 1 \\
\end{array}
\]
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

\[\begin{array}{cccccc}
5 & 6 & 2 & 4 & 7 & 8 \\
2 & 1 & 1 & 1 & 0 & 0 \\
\end{array}\]
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

$\text{runs}(\sigma) = 5,6 | 2,4,7,8 | 1,3$

a domino acts on all the dominoes on its right in its run
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6, 2, 4, 7, 8, 1, 3 \]

a domino acts on all the dominoes on its right in its run and all smaller cars in the previous run
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5,6 | 2,4,7,8 | 1,3 \]

\[ \begin{array}{c|cccc}
5 & 6 & 2 & 4 & 7 \\
2 & 1 & 1 & 1 & 1 \\
\end{array} \quad \begin{array}{c|cccc}
1 & 8 & 2 & 6 & 3 \\
0 & 1 & 1 & 2 & 0 \\
\end{array} \]

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

\[
\begin{array}{cccccccc}
5 & 6 & 2 & 4 & 7 & 8 & 1 & 3 \\
2 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{array}
\]

a domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

a domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

\[
\begin{array}{cccccc}
5 & 6 & 2 & 4 & 7 & 8 \\
2 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 3 & 0 & 0 & 0 & 0
\end{array}
\]

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6|2, 4, 7, 8|1, 3 \]

\[
\begin{array}{cccc}
5 & 6 & 2 & 4 \\
2 & 1 & 1 & 1 \\
\end{array} \quad \begin{array}{cccc}
1 & 8 & 2 & 6 \\
3 & 4 & 5 & 7 \\
0 & 1 & 1 & 2 \\
0 & 1 & 2 & 1 \\
\end{array}
\]

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run, making a schedule of actions.

\[
\begin{array}{cccc}
8 & 1 & 1 & 3 \\
1 & 0 & 0 & 0 \\
7 & 1 & & \\
\end{array}
\]

Saturday, January 13, 18
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

runs(σ) = 5, 6 | 2, 4, 7, 8 | 1, 3

a domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[\text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3\]

\[
\begin{array}{cccccc}
5 & 6 & 2 & 4 & 7 & 8 \\
2 & 1 & 1 & 1 & 1 & 1 \\
1 & 3 & 0 & 0 & 0 & 0
\end{array}
\]

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run. Make a schedule of actions.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5,6 \mid 2,4,7,8 \mid 1,3 \]

a domino acts on all the dominoes on its right in its run and all smaller cars in the previous run
make a schedule of actions
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

```
5  6  2  4  7  8  1  3
2  1  1  1  1  1  0  0
```

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

\[
\begin{array}{cccc}
5 & 6 & 2 & 1 \\
2 & 4 & 7 & 1 \\
1 & 1 & 1 & 1 \\
1 & 3 & 0 & 0 \\
\end{array}
\]

a domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[
\text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3
\]

\[
\begin{array}{ccccccc}
5 & 6 & 2 & 4 & 7 & 8 & 1 \\
2 & 2 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

a domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions

Saturday, January 13, 18
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 \quad 2, 4, 7, 8 \quad 1, 3 \]

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run. Make a schedule of actions.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

A domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

\[
\begin{array}{c|cccc}
5 & 6 & 2 & 4 & 7 \\
2 & 2 & 1 & 1 & 1 \\
\hline
1 & 3 & 0 & 0 & 0
\end{array}
\]

\[
\begin{array}{cccc}
1 & 8 & 2 & 6 \\
3 & 4 & 5 & 7 \\
0 & 1 & 1 & 2 \\
0 & 1 & 2 & 1
\end{array}
\]

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run. Make a schedule of actions.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

\[
\begin{array}{c|c|c|c|c|c|c}
5 & 6 & 2 & 4 & 7 & 8 & 1 \\hline
2 & 2 & 1 & 1 & 1 & 1 & 0 \\hline
\end{array}
\]

The algorithm constructs a tree

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.
**PARKING FUNCTIONS**
**WITH PRESCRIBED DIAGONAL CARS**

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

- A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.

**THE ALGORITHM CONSTRUCTS A TREE**
- First depict the two children of the root.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

```
    5 2
    6 2
    2 1
```

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.

```
  8 1 3 0 0
  7 8 1 3 0
  1 1 0 0
```

The algorithm constructs a tree first depict the two children of the root then insert each domino immediately to the right.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

a domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right
of each domino on which it acts
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

\[
\begin{array}{cccccccc}
5 & 6 & | & 2 & 4 & 7 & 8 & | & 1 & 3 \\
2 & 2 & | & 1 & 1 & 1 & 1 & | & 0 & 0 \\
\end{array}
\]

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right of each domino on which it acts.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

$$\text{runs}(\sigma) = 5,6 \mid 2,4,7,8 \mid 1,3$$

a domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right of each domino on which it acts

$$\begin{array}{c|c}
8 & 1 \\
1 & 0 \\
7 & 8 \\
1 & 1 \\
4 & 7 \\
1 & 1 \\
2 & 4 \\
1 & 1 \\
6 & 2 \\
2 & 1 \\
5 & 6 \\
2 & 1 \\
\end{array}$$

$$\begin{array}{c|c}
5 & 6 \\
2 & 1 \\
4 & 7 \\
1 & 1 \\
2 & 4 \\
2 & 1 \\
\end{array}$$

$$\begin{array}{c|c|c}
1 & 8 & 2 \\
6 & 3 & 4 \\
0 & 1 & 2 \\
\end{array}$$

$$\begin{array}{c|c|c|c|c}
7 & 5 & 4 & 3 \\
6 & 2 & 1 & 0 \\
8 & 7 & 6 & 5 \\
1 & 0 & 0 & 0 \\
\end{array}$$

$$q \quad 1$$
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[\text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3\]

\[
\begin{array}{cccccc}
5 & 6 & 2 & 4 & 7 & 8 \\
2 & 2 & 1 & 1 & 1 & 1 \\
\end{array}
\]

a domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right
of each domino on which it acts

\[
\begin{array}{cccc}
8 & 1 & 3 & 0 \\
7 & 8 & 1 & 3 \\
4 & 7 & 8 & 1 & 3 \\
2 & 4 & 7 & 8 & 1 \\
6 & 2 & 4 & 7 & 8 \\
5 & 6 & 2 & 4 & 7 \\
2 & 2 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c}
1 \\
3 & 1 \\
0 & 0 \\
\end{array}
\]

\[q\]

Saturday, January 13, 18
**PARKING FUNCTIONS**
**WITH PRESCRIBED DIAGONAL CARS**

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

<table>
<thead>
<tr>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>2</th>
<th>4</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>1</th>
<th>3</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.

**THE ALGORITHM CONSTRUCTS A TREE**

First depict the two children of the root, then insert each domino immediately to the right of each domino on which it acts.

\[ q \quad 1 \]

\[ 1 \quad 3 \quad 0 \quad 0 \]

\[ 3 \quad 1 \quad 0 \quad 0 \]
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

$$\text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3$$

$$\begin{array}{cccccc}
5 & 6 & 2 & 4 & 7 & 8 \\
2 & 2 & 1 & 1 & 1 & 1 \\
\end{array}$$

a domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right
of each domino on which it acts
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5,6 \mid 2,4,7,8 \mid 1,3 \]

\[
\begin{array}{ccccccc}
5 & 6 & 2 & 4 & 7 & 8 & 1 \\
2 & 2 & 1 & 1 & 1 & 1 & 0 \\
\end{array}
\]

a domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right
of each domino on which it acts

\[
q \quad 1
\]

\[
\begin{array}{ccccccc}
1 & 3 & \quad \quad 3 & 1 & 0 & 0 \\
0 & 0 & \quad \quad 0 & 0 & \quad \quad 3 & 1 & 8 \quad 0 & 0 \\
\end{array}
\]
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

\[
\begin{align*}
5 & \quad 6 \\
2 & \quad 2 \\
\end{align*}
\mid
\begin{align*}
2 & \quad 4 \\
1 & \quad 1 \\
7 & \quad 8 \\
8 & \quad 1 \\
1 & \quad 3 \\
\end{align*}
\mid
\begin{align*}
1 & \quad 3 \\
0 & \quad 0 \\
0 & \quad 0 \\
\end{align*}

a domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right
of each domino on which it acts
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

\[
\begin{array}{cccc|ccc}
5 & 6 & | & 2 & 4 & 7 & 8 \\
2 & 2 & | & 1 & 1 & 1 & 1 \\
\end{array}
\]

and all smaller cars in the previous run
make a schedule of actions

a domino acts on all the dominoes on its right in its run

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right
of each domino on which it acts
A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right of each domino on which it acts.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[
\text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3
\]

\[
\begin{array}{c|cccc|c|cccc}
5 & 6 & 2 & 4 & 7 & 8 & 1 & 3 \\
2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{array}
\]

a domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE

first depict the two children of the root
then insert each domino immediately to the right of each domino on which it acts
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

\[
egin{array}{ccccccc}
5 & 6 & | & 2 & 4 & 7 & 8 & | & 1 & 3 \\
2 & 2 & & 1 & 1 & 1 & 1 & & 0 & 0
\end{array}
\]

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right of each domino on which it acts.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5,6 \mid 2,4,7,8 \mid 1,3 \]

\[
\begin{array}{c|c|c|c|c|c|c|c}
5 & 6 & 2 & 4 & 7 & 8 & 1 & 3 \\
2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{array}
\]

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.

THE ALGORITHM CONSTRUCTS A TREE

First depict the two children of the root then insert each domino immediately to the right of each domino on which it acts.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

\[
\begin{array}{cccccccc}
5 & 6 & 2 & 4 & 7 & 8 & 1 & 3 \\
2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{array}
\]

a domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root then insert each domino immediately to the right of each domino on which it acts
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5,6 \mid 2,4,7,8 \mid 1,3 \]

\[
\begin{array}{cccccc}
5 & 6 & 2 & 4 & 7 & 8 \\
2 & 2 & 1 & 1 & 1 & 1 \\
1 & 3 & 0 & 0 & 0 & 0
\end{array}
\]

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.

**THE ALGORITHM CONSTRUCTS A TREE**

First depict the two children of the root, then insert each domino immediately to the right of each domino on which it acts.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

a domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root then insert each domino immediately to the right of each domino on which it acts

THE LEAVES OF THIS TREE
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

\[
\begin{array}{c|cccc}
5 & 6 & 2 & 4 & 7 \\
2 & 2 & 1 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 0
\end{array}
\]

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.

THE ALGORITHM CONSTRUCTS A TREE

First depict the two children of the root then insert each domino immediately to the right of each domino on which it acts.

THE LEAVES OF THIS TREE ARE THE DESIRED PARKING FUNCTIONS.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

$$\text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3$$

\[
\begin{array}{c|c|c|c|c|c|c|c}
5 & 6 & 2 & 4 & 7 & 8 & 1 & 3 \\
2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{array}
\]

a domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right of each domino on which it acts

THE LEAVES OF THIS TREE ARE THE DESIRED PARKING FUNCTIONS
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5,6|2,4,7,8|1,3 \]

\[
\begin{array}{c|cccc|c}
5 & 6 & 2 & 4 & 7 & 8 & 1 & 3 \\
2 & 2 & 1 & 1 & 1 & 1 & 0 & 0
\end{array}
\]

a domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.

THE ALGORITHM CONSTRUCTS A TREE

first depict the two children of the root then insert each domino immediately to the right of each domino on which it acts.

THE LEAVES OF THIS TREE ARE THE DESIRED PARKING FUNCTIONS
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5,6 \mid 2,4,7,8 \mid 1,3 \]

\[
\begin{array}{cccccccc}
5 & 6 & 2 & 4 & 7 & 8 & 1 & 3 \\
2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{array}
\]

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.

THE ALGORITHM CONSTRUCTS A TREE

First depict the two children of the root, then insert each domino immediately to the right of each domino on which it acts.

THE LEAVES OF THIS TREE ARE THE DESIRED PARKING FUNCTIONS.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[
\text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3
\]

| 5 \ 6 | 2 \ 4 \ 7 \ 8 | 1 \ 3 |
| 2 \ 2 | 1 \ 1 \ 1 | 0 \ 0 |

A domino acts on all the dominoes on its right in its run and all smaller cars in the previous run make a schedule of actions.

THE ALGORITHM CONSTRUCTS A TREE

- First depict the two children of the root.
- Then insert each domino immediately to the right of each domino on which it acts.

THE LEAVES OF THIS TREE ARE THE DESIRED PARKING FUNCTIONS.
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5,6 | 2,4,7,8 | 1,3 \]

\[
\begin{array}{c|c|c|c|c|c}
5 & 6 & 2 & 4 & 7 & 8 \\
2 & 2 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
1 & 8 & 2 & 6 & 3 & 4 & 5 & 7 \\
0 & 1 & 1 & 2 & 0 & 1 & 2 & 1 \\
\end{array}
\]

A domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE

First depict the two children of the root
then insert each domino immediately to the right
of each domino on which it acts

THE LEAVES OF THIS TREE
ARE THE DESIRED PARKING FUNCTIONS
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

\[
\begin{array}{c|c|c|c|c|c|c}
5 & 6 & | & 2 & 4 & 7 & 8 \\
2 & 2 & | & 1 & 1 & 1 & 1 \\
\end{array}
\]

a domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right
of each domino on which it acts

THE LEAVES OF THIS TREE
ARE THE DESIRED PARKING FUNCTIONS
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[
\text{runs}(\sigma) = 5, 6 \| 2, 4, 7, 8 \| 1, 3
\]

\[
\begin{array}{cccccc}
5 & 6 & 2 & 4 & 7 & 8 \\
2 & 2 & 1 & 1 & 1 & 1 \\
1 & 3 & 0 & 0 & 0 & 0
\end{array}
\]

A domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions

**THE ALGORITHM CONSTRUCTS A TREE**

First depict the two children of the root
then insert each domino immediately to the right
of each domino on which it acts

**THE LEAVES OF THIS TREE**

ARE THE DESIRED PARKING FUNCTIONS
PARKING FUNCTIONS
WITH PRESCRIBED DIAGONAL CARS

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

\[ \begin{array}{cccccc}
5 & 6 & 2 & 4 & 7 & 8 \\
2 & 2 & 1 & 1 & 1 & 1 \\
\end{array} \]

a domino acts on all the dominoes on its right in its run
and all smaller cars in the previous run
make a schedule of actions

THE ALGORITHM CONSTRUCTS A TREE
first depict the two children of the root
then insert each domino immediately to the right
of each domino on which it acts

THE LEAVES OF THIS TREE
ARE THE DESIRED PARKING FUNCTIONS
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE
1) starting from the original runs
1) starting from the original runs

<table>
<thead>
<tr>
<th>5</th>
<th>6</th>
<th>2</th>
<th>2</th>
<th>2</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
</table>

\[
\begin{array}{cccccccc}
8 & 1 & 3 & 0 & 0 \\
7 & 8 & 1 & 3 & 0 & 0 \\
4 & 7 & 8 & 1 & 3 & 0 & 0 \\
2 & 4 & 7 & 8 & 1 & 3 & 0 & 0 \\
6 & 2 & 4 & 1 & 1 \\
5 & 2 & 2 & 4 & 1 & 1 \\
\end{array}
\]
1) starting from the original runs

```plaintext
<table>
<thead>
<tr>
<th>5</th>
<th>6</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
```

```
<table>
<thead>
<tr>
<th>8</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
```
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

1) starting from the original runs

```
5  6  |  2  4  7  8  |  1  3
2  2  |  1  1  1  1  |  0  0
```

```
8  |  1  3
1  |  0  0
7  |  8  1  3
1  |  1  0  0
4  |  7  8  1  3
1  |  1  1  0  0
2  |  4  7  8  1
1  |  1  1  1  0
6  |  2  4
2  |  1  1
5  |  2  6  2  4
```
1) starting from the original runs
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

1) starting from the original runs

```
<table>
<thead>
<tr>
<th>5 6</th>
<th>2 4 7 8</th>
<th>1 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1 1</td>
<td>0</td>
</tr>
</tbody>
</table>
```

```
<table>
<thead>
<tr>
<th>8</th>
<th>1 3 0 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>8 1 3 0</td>
</tr>
<tr>
<td>1</td>
<td>1 0 0</td>
</tr>
</tbody>
</table>
```

```
<table>
<thead>
<tr>
<th>4</th>
<th>7 8 1 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 0 0</td>
</tr>
</tbody>
</table>
```

```
<table>
<thead>
<tr>
<th>2</th>
<th>4 7 8 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 1 0</td>
</tr>
</tbody>
</table>
```

```
<table>
<thead>
<tr>
<th>6</th>
<th>2 4 1 1</th>
</tr>
</thead>
</table>
```

```
<table>
<thead>
<tr>
<th>5</th>
<th>6 2 4 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2 1 1</td>
</tr>
</tbody>
</table>
```
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

1) starting from the original runs

\[
\begin{array}{c|cccc|c}
5 & 6 & 2 & 1 & 3 & 1 \\
2 & 1 & 1 & 1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
q & 1 & 3 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
8 & 1 & 3 & 3 & 0 \\
7 & 8 & 1 & 0 & 0 \\
4 & 7 & 8 & 1 & 3 \\
1 & 1 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
6 & 2 & 4 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
5 & 6 & 2 & 1 & 1 \\
2 & 1 & 1 & 0 & 0 \\
\end{array}
\]
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

1) starting from the original runs

\[
\begin{array}{c|c|c|c|c|c}
5 & 6 & 2 & 2 & 1 & 3 \\
2 & 4 & 7 & 8 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
1 & 3 & 8 & 1 & 3 \\
0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
3 & 1 & 7 & 8 & 1 & 3 \\
0 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
4 & 7 & 8 & 1 & 3 \\
1 & 1 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
2 & 4 & 7 & 8 & 1 & 3 \\
1 & 1 & 1 & 1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
6 & 2 & 4 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
5 & 6 & 2 & 2 & 4 \\
2 & 2 & 1 & 1 & 1 \\
\end{array}
\]
1) starting from the original runs
1) starting from the original runs

THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

1) starting from the original runs

1 3
0 0

q

1

3
0

1

3
1

0
0

1

3
0

1

3
1

8
1

0
0

1 3
1 0

0
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

1) starting from the original runs

```
5 6 | 2 4 7 8 | 1 3
2 2 | 1 1 1 1 | 0 0

q   1

1   3
0   0

3 8 | 1 1
0 1 | 0 0

q   1

3 1 8
0 0 1

8 1 3
1 0 0

7 8 1 3
1 1 0 0

4 7 8 1 3
1 1 0 0

2 4 7 8 1 3
1 1 0 0

6 2 4 7 8 1 3
1 1 0 0

5 2 6 2 4 7 8 1 3
1 1 0 0
```
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

1) starting from the original runs
1) starting from the original runs
1) starting from the original runs
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

1) starting from the original runs

```
<table>
<thead>
<tr>
<th>5 6 2 2</th>
<th>2 4 7 8 1 3 0 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0 1</td>
<td>3 0</td>
</tr>
<tr>
<td>1 0 1</td>
<td>3 8</td>
</tr>
<tr>
<td>3 0</td>
<td>1 8</td>
</tr>
<tr>
<td>0 1 0</td>
<td>0 1</td>
</tr>
</tbody>
</table>
```

Saturday, January 13, 18
1) starting from the original runs
1) starting from the original runs
1) starting from the original runs

THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

1) starting from the original runs
1) starting from the original runs

\[ q^{dinv(PF)} = q^6 \]
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

1) starting from the original runs

\[ q^{dinv(PF)} = q^6 \]
1) starting from the original runs

\[
q^{dinv(PF)} = q^6
\]
1) starting from the original runs

\begin{align*}
\begin{array}{c|c|c|c|c|c}
5 & 6 & 2 & 2 & 1 & 1111 \\
\hline
2 & 4 & 7 & 8 & 1 & 111 \\
\end{array}
\end{align*}

The algorithm that constructs our example

\[
q^{\text{dinv}(PF)} = q^6
\]

Total PF leaves

\[
2 \times 2 \cdot 3 \cdot 4 \cdot 4 \cdot 2 \cdot 3 = 1,152
\]
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

1) starting from the original runs

\[ 2 \times 2 \cdot 3 \cdot 4 \cdot 4 \cdot 2 \cdot 3 = 1,152 \]

Total PF leaves

\[ q^{\text{dinv}(PF)} = q^6 \]

Total 8 \times 8 Parking Functions

4,782,969
THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

1) starting from the original runs

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]

\[ q \]
1) starting from the original runs

\[ \begin{array}{c|c|c|c|c|c} \hline 5 & 6 & 2 & 4 & 7 & 8 \\ \hline 2 & 2 & 1 & 1 & 1 & 1 \\ \hline 1 & 3 & 0 & 0 & \hline \end{array} \]

\[ \begin{array}{c|c|c|c|c|c} \hline 8 & 1 & 3 & 4 & 7 & 8 \\ \hline 7 & 8 & 1 & 3 & 4 & 7 \\ \hline 1 & 1 & 1 & 1 & 1 & 0 \\ \hline \end{array} \]

THE ALGORITHM THAT CONSTRUCTS OUR EXAMPLE

\[ 2 \times 2 \times 3 \times 4 \times 4 \times 2 \times 3 = 1,152 \]

Total PF leaves

\[ q^{\text{dinv}}(PF) = q^6 \]

4,782,969

Total 8 \times 8 Parking Functions
1) starting from the original runs

\[
\begin{array}{c}
\begin{array}{cccc}
5 & 6 & 2 & 2 \\
2 & 1 & 1 & 1 \\
1 & 3 & 0 & 0 \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
q
\end{array}
\]

\[
\begin{array}{c}
1
\end{array}
\]

\[
\begin{array}{c}
3
\end{array}
\]

\[
\begin{array}{c}
0
\end{array}
\]

\[
\begin{array}{c}
q
\end{array}
\]

\[
\begin{array}{c}
1
\end{array}
\]

\[
\begin{array}{c}
3
\end{array}
\]

\[
\begin{array}{c}
0
\end{array}
\]

\[
\begin{array}{c}
q
\end{array}
\]

\[
\begin{array}{c}
1
\end{array}
\]

\[
\begin{array}{c}
3
\end{array}
\]

\[
\begin{array}{c}
1
\end{array}
\]

\[
\begin{array}{c}
q
\end{array}
\]

\[
\begin{array}{c}
1
\end{array}
\]

\[
\begin{array}{c}
3
\end{array}
\]

\[
\begin{array}{c}
1
\end{array}
\]

2 \times 2 \cdot 3 \cdot 4 \cdot 4 \cdot 2 \cdot 3 = 1,152

Total PF leaves

\[
q^{dinv(PF)} = q^6
\]

4,782,969
Total 8 \times 8 Parking Functions
1) starting from the original runs

\[ q \]

\[ 2 \times 2 \cdot 3 \cdot 4 \cdot 4 \cdot 2 \cdot 3 = 1,152 \]

Total PF leaves

\[ q^{\text{dinv}(PF)} = q^6 \]

4,782,969

Total 8 \times 8 Parking Functions
The composition of a Parking Function
The composition of a Parking Function
The composition of a Parking Function
The composition of a Parking Function
The composition of a Parking Function
The composition of a Parking Function
The composition of a Parking Function
The composition of a Parking Function
The composition of a Parking Function
The composition of a Parking Function
The composition of a Parking Function

\[ p = (3, 2, 1, 3) \]
The composition of a Parking Function

\[\bar{p} = (3, 2, 1, 3)\]

\[C_\alpha F[X] = \left(\frac{1}{q}\right)^{a-1} F[X - \frac{1-1/q}{z}]\Omega[zX] \Big|_{z^\alpha}\]
The composition of a Parking Function

\[ p = (3, 2, 1, 3) \]

\[ C_a F[X] = \left( \frac{1}{q} \right)^{a-1} F[X - \frac{1-1/q}{z}] \Omega[zX] \bigg|_{z^a} \]

The Compositional Shuffle Conjecture
The composition of a Parking Function

\[ p = (3, 2, 1, 3) \]

\[ C_a F[X] = \left(\frac{1}{q}\right)^{a-1} F[X - \frac{1-1/q}{z}] \Omega[zX] \bigg|_{z^a} \]

The Compositional Shuffle Conjecture

\[ \nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1 = \sum_{PF(p)} t^{area(PF)} q^{dinv(PF)} F_{pides(PF)} \]
The composition of a Parking Function

\[ p = (3, 2, 1, 3) \]

\[
C_a F[X] = (\frac{1}{q})^{a-1} F[X - \frac{1-1/q}{z}] \Omega[zX] \bigg|_{z^a}
\]

The Compositional Shuffle Conjecture

\[
\nabla C_{p_1}C_{p_2} \cdots C_{p_k} 1 = \sum_{PF(p)} t^{area(PF)} q^{\text{dinv}(PF)} F_{\text{pides}(PF)}
\]
The composition of a Parking Function

\[ p = (3, 2, 1, 3) \]

The Compositional Shuffle Conjecture

\[ C_a F[X] = (\frac{1}{q})^{a-1} F[X - \frac{1-1/q}{z}] \Omega[zX] \bigg|_{z^a} \]

\[ \nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1 = \sum_{\mathcal{P} \mathcal{F}(p)} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{pides}(PF)} \]

\[ \nabla \tilde{H}[X; q, t] = T_\mu \tilde{H}[X; q, t] \]
The composition of a Parking Function

The Compositional Shuffle Conjecture

\[ p = (3, 2, 1, 3) \]

\[ C_a F[X] = (\frac{1}{q})^{a-1} F[X - \frac{1-1/q}{z}] \Omega[zX] \bigg|_{z^a} \]

\[ \nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1 = \sum_{\mathcal{PF}(p)} \text{tarea}(PF) q^{\text{dinv}(PF)} F_{\text{pides}(PF)} \]

\[ \nabla \tilde{H}[X; q, t] = T_\mu \tilde{H}[X; q, t] \]
The composition of a Parking Function

\[ p = (3, 2, 1, 3) \]

The Compositional Shuffle Conjecture

\[ C_a F[X] = \left( \frac{1}{q} \right)^{a-1} F[X - \frac{1-1/q}{z} \Omega[zX] \bigg|_{z^a} \right] \]

The Compositional Shuffle Conjecture

\[ \nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1 = \sum_{\mathcal{PF}(p)} t^{area(PF)} q^{dinv(PF)} F_{pides(PF)} \]

\[ \nabla \tilde{H}[X; q, t] = T_\mu \tilde{H}[X; q, t] \]
The composition of a Parking Function

\[ p = (3, 2, 1, 3) \]

The Compositional Shuffle Conjecture

\[ C_\alpha F[X] = (\frac{1}{q})^{\alpha - 1} F[X - \frac{1 - 1/q}{z}\Omega[zX]]_{z^\alpha} \]

The Compositional Shuffle Conjecture

\[ \nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1 = \sum_{\mathcal{PF}(p)} \text{tarea}(PF) q^{\text{dinv}(PF)} F_{pides(PF)} \]

\[ \nabla \tilde{H}[X; q, t] = T_\mu \tilde{H}[X; q, t] \]
The composition of a Parking Function

\[ p = (3, 2, 1, 3) \]

\[ C_a F[X] = (\frac{1}{q})^{a-1} F[X - \frac{1-1/q}{z}] \Omega[zX] \bigg|_{z^a} \]

The Compositional Shuffle Conjecture

\[ \nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1 = \sum_{PF(p)} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{pides}(PF)} \]

\[ \nabla \tilde{H}[X; q, t] = T_\mu \tilde{H}[X; q, t] \]
EXISTENCE OF BIJECTIONS
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a \, 1 + \nabla C_{a-1} C_{b+1} \, 1) = \nabla C_a C_b \, 1 + \nabla C_{b+1} C_{a-1} \, 1) \]
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \] (for all \( b \leq a - 1 \))
EXISTENCE OF BIJECTIONS

$q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1)$ (for all $b \leq a - 1$)
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \] (for all \( b \leq a - 1 \))

\[ q(\Pi[b a] + \Pi[a - 1 b + 1]) = \Pi[a b] + \Pi[b + 1 a - 1] \]
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \] (for all \( b \leq a - 1 \))

**Boxed Formula**

\[ q(\Pi[b \ a] + \Pi[a - 1 \ b + 1]) = \Pi[\ a \ b] + \Pi[\ b + 1 \ a - 1] \]

\[ \Pi[a \ b] = \sum_{PF \in PF(a,b)} t_{\text{area}}(PF) q^{\text{dinv}}(PF) F_{\text{pides}}(PF) \]
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a \mathbf{1} + \nabla C_{a-1} C_{b+1} \mathbf{1}) = \nabla C_a C_b \mathbf{1} + \nabla C_{b+1} C_{a-1} \mathbf{1} \] (for all \( b \leq a - 1 \))

\[ q(\Pi[b \ a] + \Pi[a-1 \ b+1]) = \Pi[a \ b] + \Pi[b+1 \ a-1] \]

\[ \Pi[a \ b] = \sum_{PF \in PF(a,b)} t_{area}(PF) q_{dinv}(PF) F_{pides}(PF) \]

A TRULY SURPRISING DISCOVERY
**EXISTENCE OF BIJECTIONS**

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \]  
(for all \( b \leq a - 1 \))

\[ q(\Pi[b \ a] + \Pi[a - 1 \ b + 1]) = \Pi[a \ b] + \Pi[b + 1 \ a - 1] \]

\[ \Pi[a \ b] = \sum_{PF \in PF(a,b)} t_{area}(PF) q_{dinv}(PF) F_{pides}(PF) \]

**A TRULY SURPRISING DISCOVERY**

\[ \text{runs}(\sigma) = 5,6|2,4,7,8|1,3 \]
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \]  \hspace{1cm} \text{(for all } b \leq a - 1) \]

\[ q(\Pi[b a] + \Pi[a-1 b+1]) = \Pi[a b] + \Pi[b + 1 a - 1] \]

\[ \Pi[a b] = \sum_{PF \in \Pi_F(a,b)} t_{area(PF)} q_{dinv(PF)} F_{pides(PF)} \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

\begin{pmatrix}
5 & 6 \\
2 & 2 \\
\end{pmatrix}

\begin{pmatrix}
2 & 4 & 7 & 8 \\
1 & 1 & 1 & 1 \\
\end{pmatrix}

\begin{pmatrix}
1 & 3 \\
0 & 0 \\
\end{pmatrix}
EXISTENCE OF BIJECTIONS

\( q(\nabla C_b C_a \, 1 + \nabla C_{a-1} C_{b+1} \, 1) = \nabla C_a C_b \, 1 + \nabla C_{b+1} C_{a-1} \, 1 ) \) (for all \( b \leq a - 1 \))

\[
q(\Pi[b \, a] + \Pi[a - 1 \, b + 1]) = \Pi[a \, b] + \Pi[b + 1 \, a - 1]
\]

\[
\Pi[a \, b] = \sum_{PF \in \mathcal{PF}(a, b)} t_{area}(PF) q^{dinv}(PF) F_{pides}(PF)
\]

A TRULY SURPRISING DISCOVERY

\[
runs(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3
\]

\[
\text{dominoes}(\sigma)
\]
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \] (for all \( b \leq a - 1 \))

\[ q(\Pi[b \ a] + \Pi[a - 1 \ b + 1]) = \Pi[ab] + \Pi[b + 1 \ a - 1] \]

\[ \Pi[ab] = \sum_{PF \in PF(a,b)} t_{\text{area}(PF)} q_{\text{dinv}(PF)} F_{\text{pides}(PF)} \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

Parking function restricted by prescribing the diagonal cars

\[
\begin{array}{cccccc}
\text{dominoes}(\sigma)
\end{array}
\]

\[
\begin{array}{cccccc}
5 & 6 & | & 2 & 4 & 7 & 8 | 1 & 3 \\
2 & 2 & | & 1 & 1 & 1 & 1 | 0 & 0 \\
\end{array}
\]
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \] (for all \( b \leq a - 1 \))

\[ q(\Pi[b\ a] + \Pi[a - 1 \ b + 1]) = \Pi[a\ b] + \Pi[b + 1 \ a - 1] \]

\[ \Pi[a\ b] = \sum_{PF \in PF(a, b)} t_{\text{area}}(PF) q_{\text{dinv}}(PF) F_{\text{pides}}(PF) \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

Parking function restricted by prescribing the diagonal cars

\[
\begin{array}{cccccccc}
1 & 8 & 2 & 6 & 3 & 4 & 5 & 7 \\
0 & 1 & 1 & 2 & 0 & 1 & 2 & 1
\end{array}
\]

\[
\begin{array}{cccccccc}
5 & 6 & & & & & \mid & 2 & 4 & 7 & 8 & \mid & 1 & 3 \\
2 & 2 & & & & & \mid & 1 & 1 & 1 & 1 & \mid & 0 & 0
\end{array}
\]
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a \mathbf{1} + \nabla C_{a-1} C_{b+1} \mathbf{1}) = \nabla C_a C_b \mathbf{1} + \nabla C_{b+1} C_{a-1} \mathbf{1} \quad \text{(for all } b \leq a - 1) \]

\[ q(\Pi[b \, a] + \Pi[a - 1 \, b + 1]) = \Pi[a \, b] + \Pi[b + 1 \, a - 1] \]

\[ \Pi[a \, b] = \sum_{PF \in PF(a, b)} t_{area}(PF) q_{dinv}(PF) F_{pides}(PF) \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 \, | \, 2, 4, 7, 8 \, | \, 1, 3 \quad \text{dominoes}(\sigma) \]

Parking function restricted by prescribing the diagonal cars
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a \mathbf{1} + \nabla C_{a-1} C_{b+1} \mathbf{1}) = \nabla C_a C_b \mathbf{1} + \nabla C_{b+1} C_{a-1} \mathbf{1} \] (for all \( b \leq a - 1 \))

\[ q(\Pi[b \ a] + \Pi[a - 1 \ b + 1]) = \Pi[a \ b] + \Pi[b + 1 \ a - 1] \]

\[ \Pi[a \ b] = \sum_{PF \in PF(a,b)} t_{\text{area}(PF)} q^{\text{dinv}(PF)} F_{pides}(PF) \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 \ | \ 2, 4, 7, 8 \ | \ 1, 3 \]

Parking function restricted by prescribing the diagonal cars

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 1 & 2 & 0 & 1 & 2 & 1
\end{array}
\]
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \] (for all \( b \leq a - 1 \))

\[ q(\Pi[ba] + \Pi[a-1b+1]) = \Pi[ab] + \Pi[b+1a-1] \]

\[ \Pi[ab] = \sum_{PF \in PF(a,b)} t_{\text{area}}(PF) q^{\text{dinv}}(PF) F_{\text{pides}}(PF) \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

Parking function restricted by prescribing the diagonal cars
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \]  
(for all \( b \leq a - 1 \))

\[ q(\Pi[ba] + \Pi[a-1b+1]) = \Pi[ab] + \Pi[b+1a-1] \]

\[ \Pi[ab] = \sum_{PF \in PF(a,b)} t_{area}(PF) q_{dinv}(PF) F_{pides}(PF) \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

Parking function restricted by prescribing the diagonal cars

\[ \text{dominoes}(\sigma) \]
EXISTENCE OF BIJECTIONS

\( q(\nabla C_b C_a \mathbf{1} + \nabla C_{a-1} C_{b+1} \mathbf{1}) = \nabla C_a C_b \mathbf{1} + \nabla C_{b+1} C_{a-1} \mathbf{1} \) (for all \( b \leq a - 1 \))

\[
q(\Pi[b \ a] + \Pi[a - 1 \ b + 1]) = \Pi[a \ b] + \Pi[b + 1 \ a - 1]
\]

\[
\Pi[a \ b] = \sum_{PF \in PF(a, b)} t_{area}(PF) q_{\text{dinv}}(PF) F_{\text{pides}}(PF)
\]

A TRULY SURPRISING DISCOVERY

\[\text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3\]

Parking function restricted by prescribing the diagonal cars
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \] (for all \( b \leq a - 1 \))

\[ q(\Pi[ba] + \Pi[a-1b+1]) = \Pi[ab] + \Pi[b+1a-1] \]

\[ \Pi[a\ b] = \sum_{PF \in PF(a,b)} t_{area(PF)} q_{dinv(PF)} F_{pides(PF)} \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

Parking function restricted by prescribing the diagonal cars

\[ \text{dominoes}(\sigma) \]
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a \mathbf{1} + \nabla C_{a-1} C_{b+1} \mathbf{1}) = \nabla C_a C_b \mathbf{1} + \nabla C_{b+1} C_{a-1} \mathbf{1} \]  (for all \( b \leq a - 1 \))

\[
q(\Pi[b a] + \Pi[a - 1 b + 1]) = \Pi[a b] + \Pi[b + 1 a - 1]
\]

\[
\Pi[a b] = \sum_{PF \in PF(a,b)} t_{\text{area}}(PF) q_{\text{dinv}}(PF) F_{\pi d e s}(PF)
\]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

Parking function restricted by prescribing the diagonal cars
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \]  
(for all \( b \leq a - 1 \))

\[ q(\Pi[ba] + \Pi[a-1b+1]) = \Pi[ab] + \Pi[b+1a-1] \]

\[ \Pi[ab] = \sum_{PF \in PF(a,b)} t_{area}(PF) q_{dinv}(PF) F_{pides}(PF) \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

Parking function restricted by prescribing the diagonal cars
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \] (for all \( b \leq a - 1 \))

\[ q(\Pi[ba] + \Pi[a-1b+1]) = \Pi[ab] + \Pi[b+1a-1] \]

\[ \Pi[ab] = \sum_{PF \in PF(a,b)} t_{\text{area}}(PF) q_{\text{dinv}}(PF) F_{\text{pides}}(PF) \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5,6 \mid 2,4,7,8 \mid 1,3 \]

Parking function restricted by prescribing the diagonal cars

satisfy exactly the same identities!!!
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \] (for all \( b \leq a - 1 \))

\[
q(\Pi[a b] + \Pi[a - 1 b + 1]) = \Pi[a b] + \Pi[b + 1 a - 1]
\]

\[
\Pi[a b] = \sum_{PF \in PF(a,b)} t_{area}(PF) q_{dinv}(PF) F_{pides}(PF)
\]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

Parking function restricted by prescribing the diagonal cars

satisfy exactly the same identities!!!
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \] (for all \( b \leq a - 1 \))

\[ q(\Pi[b \ a] + \Pi[a - 1 \ b + 1]) = \Pi[a \ b] + \Pi[b + 1 \ a - 1] \]

\[ \Pi[a \ b] = \sum_{PF \in PF(a,b)} t_{area}(PF) q_{dinv}(PF) F_{pides}(PF) \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5,6 \ | 2,4,7,8 \ | 1,3 \]

Parking function restricted by prescribing the diagonal cars

satisfy exactly the same identities!!!
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a \mathbf{1} + \nabla C_{a-1} C_{b+1} \mathbf{1}) = \nabla C_a C_b \mathbf{1} + \nabla C_{b+1} C_{a-1} \mathbf{1} \]  (for all \( b \leq a - 1 \))

\[ q(\Pi[ab] + \Pi[a-1b+1]) = \Pi[ab] + \Pi[b+1a-1] \]

\[ \Pi[ab] = \sum_{PF \in PF(a,b)} t_{area}(PF) q_{dinv}(PF) F_{pides}(PF) \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5,6 \mid 2,4,7,8 \mid 1,3 \]

Parking function restricted by prescribing the diagonal cars

satisfy exactly the same identities!!!
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \] (for all \( b \leq a - 1 \))

\[ q(\Pi[b \ a] + \Pi[a - 1 \ b + 1]) = \Pi[a \ b] + \Pi[b + 1 \ a - 1] \]

\[ \Pi[a \ b] = \sum_{PF \in PF(a, b)} t_{area}(PF) q_{dinv}(PF) F_{pides}(PF) \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 \ | \ 2, 4, 7, 8 \ | \ 1, 3 \]

Parking function restricted by prescribing the diagonal cars

satisfy exactly the same identities!!!
EXISTENCE OF BIJECTIONS

\[ q(\nabla C_b C_a 1 + \nabla C_{a-1} C_{b+1} 1) = \nabla C_a C_b 1 + \nabla C_{b+1} C_{a-1} 1 \] (for all \( b \leq a - 1 \))

\[ q(\Pi[b a] + \Pi[a - 1 b + 1]) = \Pi[a b] + \Pi[b + 1 a - 1] \]

\[ \Pi[a b] = \sum_{PF \in PF(a,b)} t_{area}(PF) q^{\text{dinv}}(PF) F_{pides}(PF) \]

A TRULY SURPRISING DISCOVERY

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]

Parking function restricted by prescribing the diagonal cars

satisfy exactly the same identities!!!
Bar Diagrams !!
Bar Diagrams !!

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]
Bar Diagrams !!

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]
Bar Diagrams!!

\[ \text{runs}(\sigma) = 5, 6 \mid 2, 4, 7, 8 \mid 1, 3 \]
Bar Diagrams!!

$\text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3$
The Bar Diagrams we present here correspond to permutations with last run of length 2.
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition.
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition. Excepted those in the last run, the length of bar $c$ is is given by the schedule.
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition. Excepted those in the last run, the length of bar c is is given by the schedule.
The Bar Diagrams we present here correspond to permutations with last run of length 2.

Thus all our PF’s have a 2 part composition.

Excepted those in the last run, the length of bar $c$ is given by the schedule.
The Bar Diagrams we present here correspond to permutations with last run of length 2.

Thus all our PF’s have a 2 part composition excepted those in the last run,
the length of bar $c$ is is given by the schedule.
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition. Excepted those in the last run, the length of bar $c$ is is given by the schedule.
Bar Diagrams!!

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

The Bar Diagrams we present here correspond to permutations with last run of length 2.

Thus all our PF’s have a 2 part composition.

Excepted those in the last run,

the length of bar \( c \) is is given by the schedule.
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition.

Excepted those in the last run, the length of bar $\mathbf{c}$ is is given by the schedule.
The Bar Diagrams we present here correspond to permutations with last run of length 2.

Thus all our PF’s have a 2 part composition excepted those in the last run,

the length of bar \( c \) is given by the schedule.
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition.

Excepted those in the last run, the length of bar $c$ is is given by the schedule.

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition. Excepted those in the last run, the length of bar $c$ is is given by the schedule.

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2.

Thus all our PF’s have a 2 part composition. Excepted those in the last run,

the length of bar \( c \) is given by the schedule.

The car labelled bar diagram corresponds to the Parking Function below.

Sat\text{urday}, \text{ January 13, 18}
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition.

Excepted those in the last run, the length of bar $c$ is given by the schedule Bar $c$ is up if and only if $c$ is on the right part.

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2.

Thus all our PF’s have a 2 part composition. Excepted those in the last run,

the length of bar c is given by the schedule.

Bar c is up if and only if c is on the right part.

Bar Diagrams start as follows.

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2

Thus all our PF’s have a 2 part composition

Excepted those in the last run,

the length of bar $c$ is given by the schedule

Bar $c$ is up if and only if $c$ is on the right part

Bar Diagrams start as follows

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition. Exptected those in the last run, the length of bar $c$ is given by the schedule. Bar $c$ is up if and only if $c$ is on the right part. Bar Diagrams start as follows:

The car labelled bar diagram corresponds to the Parking Function below.
Bar Diagrams !!

The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition. Excepted those in the last run, the length of bar $c$ is given by the schedule. Bar $c$ is up if and only if $c$ is on the right part.

Bar Diagrams start as follows:

<table>
<thead>
<tr>
<th>8</th>
<th>1</th>
<th>3</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1</td>
<td>8</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The number of yellow cells equals the number of cells that it touches, when rotated.

The car labelled bar diagram corresponds to the Parking Function below.
Bar Diagrams!!

The Bar Diagrams we present here correspond to permutations with last run of length 2.

Thus all our PF’s have a 2 part composition.

Excepted those in the last run,

the length of bar $c$ is is given by the schedule.

Bar $c$ is up if and only if $c$ is on the right part.

Bar Diagrams start as follows.

The number of yellow cells equals the number of cells that it touches, when rotated.

Up bars must be rotated counterclockwise, down bars clockwise.

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2.

Thus all our PF's have a 2 part composition. Excepted those in the last run, the length of bar $c$ is given by the schedule.

Bar $c$ is up if and only if $c$ is on the right part.

Bar Diagrams start as follows:

The number of yellow cells equals the number of cells that it touches, when rotated.

Up bars must be rotated counterclockwise, down bars clockwise.

The dinv contribution of each cell, whether in a up or down bar is a power of $q$.

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition. Exceptional those in the last run, the length of bar \( c \) is is given by the schedule. Bar \( c \) is up if and only if \( c \) is on the right part.

Bar Diagrams start as follows.

The number of yellow cells equals the number of cells that it touches, when rotated. Up bars must be rotated counterclockwise, down bars clockwise.

The dinv contribution of each cell, whether in a up or down bar is a power of \( q \).

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2.

Thus all our PF’s have a 2 part composition.

Excepted those in the last run,

the length of bar $c$ is is given by the schedule.

Bar $c$ is up if and only if $c$ is on the right part.

Bar Diagrams start as follows.

The number of yellow cells equals the number of cells that it touches, when rotated.

Up bars must be rotated counterclockwise, down bars clockwise.

The dinv contribution of each cell, whether in a up or down bar is a power of $q$.

The car labelled bar diagram corresponds to the Parking Function below.
Bar Diagrams!!

The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition.

Excepted those in the last run, the length of bar $c$ is is given by the schedule $\text{Bar } c$ is up if and only if $c$ is on the right part.

Bar Diagrams start as follows:

The number of yellow cells equals the number of cells that it touches, when rotated. Up bars must be rotated counterclockwise, down bars clockwise.

The dinv contribution of each cell, whether in a up or down bar is a power of $q$.

The car labelled bar diagram corresponds to the Parking Function below. .
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition. Excepted those in the last run, the length of bar $c$ is is given by the schedule.

Bar $c$ is up if and only if $c$ is on the right part. Bar Diagrams start as follows:

The number of yellow cells equals the number of cells that it touches, when rotated.

Up bars must be rotated counterclockwise, down bars clockwise.

The dinv contribution of each cell, whether in a up or down bar is a power of $q$.

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition. Excepted those in the last run, the length of bar $c$ is is given by the schedule. Bar $c$ is up if and only if $c$ is on the right part. Bar Diagrams start as follows:

The number of yellow cells equals the number of cells that it touches, when rotated. Up bars must be rotated counterclockwise, down bars clockwise. The dinv contribution of each cell, whether in a up or down bar is a power of $q$.

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition. Excepted those in the last run, the length of bar \( c \) is given by the schedule. Bar \( c \) is up if and only if \( c \) is on the right part.

Bar Diagrams start as follows:

The number of yellow cells equals the number of cells that it touches, when rotated. Up bars must be rotated counterclockwise, down bars clockwise.

The \( \text{dinv} \) contribution of each cell, whether in a up or down bar is a power of \( q \).

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2.

Thus all our PF’s have a 2 part composition. Excepted those in the last run, the length of bar \( c \) is given by the schedule \( \text{Bar } c \text{ is up if and only if } c \text{ is on the right part} \).

Bar Diagrams start as follows:

The number of yellow cells equals the number of cells that it touches, when rotated. Up bars must be rotated counterclockwise, down bars clockwise.

The \( \text{dinv} \) contribution of each cell, whether in a up or down bar is a power of \( q \).

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2.

Thus all our PF’s have a 2 part composition.

Excepted those in the last run, the length of bar $c$ is is given by the schedule.

Bar $c$ is up if and only if $c$ is on the right part.

Bar Diagrams start as follows.

The number of yellow cells equals the number of cells that it touches, when rotated.

Up bars must be rotated counterclockwise, down bars clockwise.

The dinv contribution of each cell, whether in a up or down bar is a power of $q$.

The car labelled bar diagram corresponds to the Parking Function below.
The Bar Diagrams we present here correspond to permutations with last run of length 2.

Thus all our PF’s have a 2 part composition.

Excepted those in the last run,

the length of bar $c$ is is given by the schedule.

Bar $c$ is up if and only if $c$ is on the right part.

Bar Diagrams start as follows.

The number of yellow cells equals the number of cells that it touches, when rotated.

Up bars must be rotated counterclockwise, down bars clockwise.

The dinv contribution of each cell, whether in a up or down bar is a power of $q$.

The car labelled bar diagram corresponds to the Parking Function below.
Bar Diagrams !!

The Bar Diagrams we present here correspond to permutations with last run of length 2. Thus all our PF’s have a 2 part composition. Excepted those in the last run, the length of bar \( c \) is given by the schedule. Bar \( c \) is up if and only if \( c \) is on the right part. Bar Diagrams start as follows:

The number of yellow cells equals the number of cells that it touches, when rotated. Up bars must be rotated counterclockwise, down bars clockwise. The dinv contribution of each cell, whether in a up or down bar is a power of \( q \). The car labelled bar diagram corresponds to the Parking Function below.
PF or BD Polynomials
$P_\sigma(x; q, F) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{\text{inv}(PF)} F_{pides}(PF)$
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{dinv(PF)} F_{pides}(PF) \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} F_{\text{pides}}(PF) \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma(x; q, 1) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} \]
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} F_{pides}(PF) \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma(x; q, 1) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]
PF or BD Polynomials

\[
P_\sigma(x; q, F) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} F_{pides}(PF)
\]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[
P_\sigma(x; q, 1) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b
\]

Where \( n \) gives the number of cars,
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} F_{pides}(PF) \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma(x; q, 1) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]

Where \( n \) gives the number of cars,

and \( A_b(q) \) is the dinv contribution of the colored \( q \) labelled
PF or BD Polynomials

\[ P_{\sigma}(x; q, F) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} F_{\text{pides}(PF)} \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[
P_{\sigma}(x; q, 1) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b
\]

Where \( n \) gives the number of cars,
and \( A_b(q) \) is the \text{dinv} contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} F_{pides(PF)} \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma(x; q, 1) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the dinv contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{dinv(PF)} F_{pides}(PF) \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma(x; q, 1) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{dinv(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the dinv contribution of the colored q labelled bar diagrams with b bars above the ground line.

\[ = q^3(1 + q)(q^2 + q^3) \]
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} F_{pides}(PF) \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma(x; q, 1) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the dinv contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.

\[ = q^3(1 + q)(q^2 + q^3) \]

The polynomial \( P_\sigma(x; q, 1) \) depends only on the lengths of the bars!!
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} F_{pides}(PF) \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma(x; q, 1) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the \( dinv \) contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.

\[ = q^3(1 + q)(q^2 + q^3) \]

The polynomial \( P_\sigma(x; q, 1) \) depends only on the lengths of the bars!!

Legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), (with \( w_i \leq w_{i-1} + 1 \)).
PF or BD Polynomials

\[
P_\sigma(x; q, F) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} F_{pides(PF)}
\]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[
P_\sigma(x; q, 1) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b
\]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the dinv contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.

\[
q^3(1 + q)(q^2 + q^3)
\]

The polynomial \( P_\sigma(x; q, 1) \) depends only on the lengths of the bars!!

Legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), (with \( w_i \leq w_{i-1} + 1 \)).

We define
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{dinv(PF)} F_{pides}(PF) \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma(x; q, 1) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{dinv(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the dinv contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.

\[ = q^3(1 + q)(q^2 + q^3) \]

Legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), (with \( w_i \leq w_{i-1} + 1 \)).

We define

\[ P_W[X_k; q] = \sum_{BD \in BD_w} m_{BD}[X_k] \pi_{BD}(q) \]

The polynomial \( P_\sigma(x; q, 1) \) depends only on the lengths of the bars!!
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} F_{\text{pides}(PF)} \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma(x; q, 1) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the dinv contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.

\[ = q^3(1 + q)(q^2 + q^3) \]

The polynomial \( P_\sigma(x; q, 1) \) depends only on the lengths of the bars!!

Legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), (with \( w_i \leq w_{i-1} + 1 \)).

We define

\[ P_W[X_k; q] = \sum_{BD \in BD_W} m_{BD}[X_k] \pi_{BD}(q) \]

with \( X_k = (x_{-1}, x_0, x_1, x_2, \ldots, x_{k-1}) \) and
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} F_{\text{pides}(PF)} \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[
P_\sigma(x; q, 1) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b
\]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the dinv contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.

\[
= q^3(1 + q)(q^2 + q^3)
\]

The polynomial \( P_\sigma(x; q, 1) \) depends only on the lengths of the bars!!

Legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), (with \( w_i \leq w_{i-1} + 1 \)).

We define

\[
P_W[X_k; q] = \sum_{BD \in \mathcal{BD}_W} m_{BD}[X_k] \pi_{BD}(q)
\]

with \( X_k = (x_{-1}, x_0, x_1, x_2, \ldots, x_{k-1}) \) and set \( Q_W(x; q) = P_W(X_k; q) \big|_{x_i=x} \)
PF or BD Polynomials

\[
P_\sigma(x; q, F) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} F_{pides}(PF)
\]

where \(r(PF)\) gives the number of cars on the right part, (the number of bars up).

\[
P_\sigma(x; q, 1) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b
\]

Where \(n\) gives the number of cars, and \(A_b(q)\) is the dinv contribution of the colored \(q\) labelled bar diagrams with \(b\) bars above the ground line.

\[
= q^3(1 + q)(q^2 + q^3)
\]

The polynomial \(P_\sigma(x; q, 1)\) depends only on the lengths of the bars!!

Legal schedules \(W = (w_1, w_2, \ldots, w_{k-1})\), (with \(w_i \leq w_{i-1} + 1\)).

We define

\[
P_W[X_k; q] = \sum_{BD \in BD_W} m_{BD}[X_k] \pi_{BD}(q)
\]

with \(X_k = (x_{-1}, x_0, x_1, x_2, \ldots, x_{k-1})\) and set \(Q_W(x; q) = P_W(X_k; q)\bigg|_{x_i = x}\)

\[
m_{BD}[X_k] = x_e \prod_{j=1}^{k-1} x_j^{\chi(\text{bar } j \text{ is up})},
\]
PF or BD Polynomials

\[ P_{\sigma}(x; q, F) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{dinv(PF)} F_{pides}(PF) \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_{\sigma}(x; q, 1) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{dinv(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the dinv contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.

\[ = q^3(1 + q)(q^2 + q^3) \]

The polynomial \( P_{\sigma}(x; q, 1) \) depends only on the lengths of the bars!!

Legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), (with \( w_i \leq w_{i-1} + 1 \)).

We define

\[ P_W[X_k; q] = \sum_{BD \in BD_W} m_{BD}[X_k] \pi_{BD}(q) \]

with \( X_k = (x_{-1}, x_0, x_1, x_2, \ldots, x_{k-1}) \) and set \( Q_W(x; q) = P_W(X_k; q) \bigg|_{x_i=x} \)

\[ m_{BD}[X_k] = x_\epsilon \prod_{j=1}^{k-1} x_j^{(\text{bar } j \text{ is up})}, \quad \pi_{BD}(q) = q^{x(x_\epsilon=x_{-1})} \sum_{LBD} \prod_{i=1}^{k-1} q^{r_i(LBD)} \]
where $r(PF)$ gives the number of cars on the right part, (the number of bars up).

$$P_\sigma(x; q, 1) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b$$

The polynomial $P_\sigma(x; q, 1)$ depends only on the lengths of the bars!!

Legal schedules $W = (w_1, w_2, \ldots, w_{k-1})$, (with $w_i \leq w_{i-1} + 1$).

We define

$$P_W[X_k; q] = \sum_{BD \in BD_W} m_{BD}[X_k] \pi_{BD}(q)$$

with $X_k = (x_{-1}, x_0, x_1, x_2, \ldots, x_{k-1})$ and set $Q_W(x; q) = P_W(X_k; q) \bigg| _{x_i=x}$
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} F_{pides}(PF) \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma(x; q, 1) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the \( dinv \) contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.

\[ = q^3(1 + q)(q^2 + q^3) \]

Legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), (with \( w_i \leq w_{i-1} + 1 \)).

We define

\[ P_W[X_k; q] = \sum_{BD \in BD_w} m_{BD}[X_k] \pi_{BD}(q) \]

with \( X_k = (x_{-1}, x_0, x_1, x_2, \ldots, x_{k-1}) \) and set \( Q_W(x; q) = P_W(X_k; q) \bigg|_{x_i=x} \)

\[ m_{BD}[X_k] = x_\epsilon \prod_{j=1}^{k-1} x_j^{\chi(\text{bar } j \text{ is up})}, \quad \pi_{BD}(q) = q^{\chi(x_\epsilon=x_{-1})} \sum_{LBD} \prod_{i=1}^{k-1} q^{r_i(LBD)} \]
PF or BD Polynomials

\[ P_\sigma (x; q, F) = \sum_{PF\in T(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} F_{\text{pides}(PF)} \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma (x; q, 1) = \sum_{PF\in T(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the dinv contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.

\[ = q^3 (1 + q)(q^2 + q^3) \]

The polynomial \( P_\sigma (x; q, 1) \) depends only on the lengths of the bars!!

Legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), (with \( w_i \leq w_{i-1} + 1 \)).

We define

\[ P_W [X_k; q] = \sum_{BD\in BD_w} m_{BD}[X_k] \pi_{BD}(q) \]

with \( X_k = (x_{-1}, x_0, x_1, x_2, \ldots, x_{k-1}) \) and set \( Q_W(x; q) = P_W(X_k; q) \bigg|_{x_i=x} \)

\[ m_{BD}[X_k] = x_\epsilon \prod_{j=1}^{k-1} x_j^{(\text{bar } j \text{ is up})}, \quad \pi_{BD}(q) = q^{x (x_\epsilon = x_{-1})} \sum_{LBD} \prod_{i=1}^{k-1} q^{r_i(LBD)} \]
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{dinv(PF)} F_{\text{pides}}(PF) \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma(x; q, 1) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{dinv(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the dinv contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.

\[ = q^3(1 + q)(q^2 + q^3) \]

The polynomial \( P_\sigma(x; q, 1) \) depends only on the lengths of the bars!!

Legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), (with \( w_i \leq w_{i-1} + 1 \)).

We define

\[ P_W[X_k; q] = \sum_{BD \in \mathcal{BD}_W} m_{BD}[X_k] \pi_{BD}(q) \]

with \( X_k = (x_{-1}, x_0, x_1, x_2, \ldots, x_{k-1}) \) and set \( Q_W(x; q) = P_W(X_k; q) \bigg|_{x_i=x} \)

\[ m_{BD}[X_k] = x_\epsilon \prod_{j=1}^{k-1} x_j^{(\text{bar \ } j \text{ is up})}, \quad \pi_{BD}(q) = q^{\chi(x_\epsilon=x_{-1})} \sum_{\mathcal{LBD}} \prod_{i=1}^{k-1} q^{r_i(\mathcal{LBD})} \]
PF or BD Polynomials

\[
P_\sigma(x; q, F) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} F_{pides}(PF)
\]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[
P_\sigma(x; q, 1) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{dinv(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b
\]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the dinv contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.

\[
q^3(1 + q)(q^2 + q^3)
\]

The polynomial \( P_\sigma(x; q, 1) \) depends only on the lengths of the bars!!

Legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), (with \( w_i \leq w_{i-1} + 1 \)).

We define

\[
P_W[X_k; q] = \sum_{BD \in BD_W} m_{BD}[X_k] \pi_{BD}(q)
\]

with \( X_k = (x_{-1}, x_0, x_1, x_2, \ldots, x_{k-1}) \) and set \( Q_W(x; q) = P_W(X_k; q) \bigg|_{x_i=x} \)

\[
m_{BD}[X_k] = x_\epsilon \prod_{j=1}^{k-1} x_j^{\chi(\text{bar } j \text{ is up})}, \quad \pi_{BD}(q) = q^{\chi(x_\epsilon=x_{-1})} \sum_{LBD} \prod_{i=1}^{k-1} q^{r_i(LBD)}
\]
PF or BD Polynomials

\[ P_\sigma(x; q, F) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} F_{\text{pides}(PF)} \]

where \( r(PF) \) gives the number of cars on the right part, (the number of bars up).

\[ P_\sigma(x; q, 1) = \sum_{PF \in \mathcal{T}(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b \]

Where \( n \) gives the number of cars, and \( A_b(q) \) is the \( \text{dinv} \) contribution of the colored \( q \) labelled bar diagrams with \( b \) bars above the ground line.

\[ = q^3(1 + q)(q^2 + q^3) \]

The polynomial \( P_\sigma(x; q, 1) \) depends only on the lengths of the bars!!

Legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), (with \( w_i \leq w_{i-1} + 1 \)).

We define

\[ P_W[X_k; q] = \sum_{BD \in BD_W} m_{BD}[X_k] \pi_{BD}(q) \]

with \( X_k = (x_{-1}, x_0, x_1, x_2, \ldots, x_{k-1}) \) and set \( Q_W(x; q) = P_W(X_k; q) \bigg|_{x_i = x} \)

\[ m_{BD}[X_k] = x_{-1} \prod_{j=1}^{k-1} x_j^{(\text{bar} \ j \ i s \ u p)}, \quad \pi_{BD}(q) = q^{\chi(x_{-1} = x_0)} \sum_{LBD} \prod_{i=1}^{k-1} q^{r_i(LBD)} \]
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem
Conjugation of this remarkable polynomial

Theorem

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem

For any schedule $W = (w_1, w_2, \ldots, w_{k-1})$ and any $1 \leq w \leq w_{k-1} + 1$ we have

$$P_{W,w}[X_{k+1}; q] = \frac{x_k - q^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_{k-i} \to qx_{k-i}; 1 \leq i \leq w}$$
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem

For any schedule $W = (w_1, w_2, \ldots, w_{k-1})$ and any $1 \leq w \leq w_{k-1} + 1$ we have

$$P_{W,w}[X_{k+1};q] = \frac{x_k - q^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_{k-i} \rightarrow qx_{k-i}; 1 \leq i \leq w}$$

$$P_2[x_{-1}, x_0, x_1]; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1$$
Theorem

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W,w}[X_{k+1}; q] = \frac{x_k - q^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q]
\]

\[\left. \right|_{x_{k-i} \rightarrow qx_{k-i}; 1 \leq i \leq w}

Proof

\[
P_2[x_{-1}, x_0, x_1; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1
\]
Theorem

For any schedule $W = (w_1, w_2, \ldots, w_{k-1})$ and any $1 \leq w \leq w_{k-1} + 1$ we have

$$P_{W,w}[X_{k+1};q] = \frac{x_k-q^w}{1-q} P_W[X_k;q] + \frac{1-x_k}{1-q} P_W[X_k;q] \bigg|_{x_{k-i} \rightarrow qx_{k-i}; 1 \leq i \leq w}$$

Proof

$$P_2[x_{-1}, x_0, x_1; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1$$

Suppose that $m_{BD}[X_k] \pi_{BD}(q)$ is a summand of $P_W[X_k;q]$
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem

For any schedule $W = (w_1, w_2, \ldots, w_{k-1})$ and any $1 \leq w \leq w_{k-1} + 1$ we have

$$P_{W,w}[X_{k+1};q] = \frac{x_k-q^w}{1-q} P_W[X_k;q] + \frac{1-x_k}{1-q} P_W[X_k;q] \bigg|_{x_{k-i} \rightarrow qx_{k-i}; 1 \leq i \leq w}$$

Proof

$P_2[x_{-1}, x_0, x_1]; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1$

Suppose that $m_{BD}[X_k]\pi_{BD}(q)$ is a summand of $P_W[X_k;q]$.

Suppose that exactly $a$ of the variables $x_{i-1}, \ldots, x_{i-w}$ are in the monomial $m_{BD}[X_k]$. 

Saturday, January 13, 18
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

**Theorem**

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W, w}[X_{k+1}; q] = \frac{x_k - q^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_{k-i} \rightarrow qx_{k-i}} \; ; 1 \leq i \leq w
\]

**Proof**

\[ P_2[x_{-1}, x_0, x_1]; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1 \]

Suppose that \( m_{BD}[X_k] \pi_{BD}(q) \) is a summand of \( P_W[X_k; q] \)

Suppose that exactly \( a \) of the variables \( x_{i-1}, \ldots, x_{i-w} \) are in the monomial \( m_{BD}[X_k] \).

This given, we can obtain two of the properly colored bar diagrams of the schedule \( W, w \)
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W,w}[X_{k+1}; q] = \frac{x_k - q^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_{k-i} \to qx_{k-i}; 1 \leq i \leq w}
\]

Proof

\[
P_2[x_{-1}, x_0, x_1]; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1
\]

Suppose that \( m_{BD}[X_k] \pi_{BD}(q) \) is a summand of \( P_W[X_k; q] \).

Suppose that exactly \( a \) of the variables \( x_{i-1}, \ldots, x_{i-w} \) are in the monomial \( m_{BD}[X_k] \).

This given, we can obtain two of the properly colored bar diagrams of the schedule \( W, w \) by appending to \( BD \) first an up bar of length \( w \) with \( w - a \) red upper cells.
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem

For any schedule $W = (w_1, w_2, \ldots, w_{k-1})$ and any $1 \leq w \leq w_{k-1} + 1$ we have

$$P_{W,w}[X_{k+1}; q] = \frac{x_k - q^w}{1 - q} P_W[X_k; q] + \frac{1 - x_k}{1 - q} P_W[X_k; q] \bigg|_{x_k \rightarrow qx_{k-i}; 1 \leq i \leq w}$$

Proof

$$P_2[x_{-1}, x_0, x_1]; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1$$

Suppose that $m_{BD}[X_k] \pi_{BD}(q)$ is a summand of $P_W[X_k; q]$. Suppose that exactly $a$ of the variables $x_{i-1}, \ldots, x_{i-w}$ are in the monomial $m_{BD}[X_k]$. This given, we can obtain two of the properly colored bar diagrams of the schedule $W, w$ by appending to $BD$ first an up bar of length $w$ with $w - a$ red upper cells and then appending to $BD$ a down bar of length $w$ with $a$ red lower cells.
Theorem

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W,w}[X_{k+1};q] = \frac{x_k-q^w}{1-q} P_W[X_k;q] + \frac{1-x_k}{1-q} P_W[X_k;q] \bigg|_{x_{k-i} \rightarrow qx_{k-i}; 1 \leq i \leq w}
\]

Proof

\[ P_2[x_{-1}, x_0, x_1]; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1 \]

Suppose that \( m_{BD}[X_k] \pi_{BD}(q) \) is a summand of \( P_W[X_k;q] \).

Suppose that exactly \( a \) of the variables \( x_{i-1}, \ldots, x_{i-w} \) are in the monomial \( m_{BD}[X_k] \).

This given, we can obtain two of the properly colored bar diagrams of the schedule \( W, w \) by appending to \( BD \) first an up bar of length \( w \) with \( w - a \) red upper cells

and then appending to \( BD \) a down bar of length \( w \) with \( a \) red lower cells.

Calling \( BD^{(1)} \) and \( BD^{(2)} \) the resulting colored bar diagrams of \( W, w \).
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem
For any schedule $W = (w_1, w_2, \ldots, w_{k-1})$ and any $1 \leq w \leq w_{k-1} + 1$ we have

$$P_{W,w}[X_{k+1};q] = \frac{x_k - q^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_{k-i} \rightarrow qx_{k-i}; 1 \leq i \leq w}$$

Proof

$$P_2[x_{-1}, x_0, x_1]; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1$$

Suppose that $m_{BD}[X_k] \pi_{BD}(q)$ is a summand of $P_W[X_k; q]$

Suppose that exactly $a$ of the variables $x_{i-1}, \ldots, x_{i-w}$ are in the monomial $m_{BD}[X_k]$.

This given, we can obtain two of the properly colored bar diagrams of the schedule $W, w$

by appending to $BD$ first an up bar of length $w$ with $w - a$ red upper cells

and then appending to $BD$ a down bar of length $w$ with $a$ red lower cells.

Calling $BD^{(1)}$ and $BD^{(2)}$ the resulting colored bar diagrams of $W, w$.

$$m_{BD^{(1)}} \pi_{BD^{(1)}} = x_k m_{BD} \pi_{BD} (1 + \cdots + q^{a-1}) = m_{BD} \pi_{BD} \frac{x_k - x_k q^a}{1-q}$$
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

**Theorem**

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W,w}[X_{k+1};q] = \frac{x_k-q^w}{1-q} P_W[X_k;q] + \frac{1-x_k}{1-q} P_W[X_k;q] \bigg|_{x_{k-i} \rightarrow qx_{k-i}} ; 1 \leq i \leq w
\]

**Proof**

\( P_2[x_{-1}, x_0, x_1];q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1 \)

Suppose that \( m_{BD}[X_k] \pi_{BD}(q) \) is a summand of \( P_W[X_k;q] \)

Suppose that exactly \( a \) of the variables \( x_{i-1}, \ldots, x_{i-w} \) are in the monomial \( m_{BD}[X_k] \).

This given, we can obtain two of the properly colored bar diagrams of the schedule \( W, w \) by appending to \( BD \) first an up bar of length \( w \) with \( w - a \) red upper cells and then appending to \( BD \) a down bar of length \( w \) with \( a \) red lower cells.

Calling \( BD^{(1)} \) and \( BD^{(2)} \) the resulting colored bar diagrams of \( W, w \).

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} = x_k m_{BD} \pi_{BD}(1 + \cdots + q^{a-1}) = m_{BD} \pi_{BD} \frac{x_k-q^a}{1-q}
\]

\[
m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD}(q^a + \cdots + q^{w-1}) = m_{BD} \pi_{BD} \frac{q^a-q^w}{1-q}.
\]

Saturday, January 13, 18
Theorem

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W,w}[X_{k+1}; q] = \frac{x_k - q^w}{1-q} P_W[X_k; q] + \frac{1 - x_k}{1-q} P_W[X_k; q] \bigg|_{x_k-i \rightarrow qx_k-i}; 1 \leq i \leq w
\]

Proof

\( P_2[x_{-1}, x_0, x_1]; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1 \)

Suppose that \( m_{BD}[X_k] \pi_{BD}(q) \) is a summand of \( P_W[X_k; q] \).

Suppose that exactly \( a \) of the variables \( x_{i-1}, \ldots, x_{i-w} \) are in the monomial \( m_{BD}[X_k] \).

This given, we can obtain two of the properly colored bar diagrams of the schedule \( W, w \) by appending to \( BD \) first an up bar of length \( w \) with \( w-a \) red upper cells

and then appending to \( BD \) a down bar of length \( w \) with \( a \) red lower cells.

Calling \( BD^{(1)} \) and \( BD^{(2)} \) the resulting colored bar diagrams of \( W, w \).

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} = x_k m_{BD} \pi_{BD} (1 + \cdots + q^{a-1}) = m_{BD} \pi_{BD} \frac{x_k - x_k q^a}{1-q}
\]

\[
m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD} (q^a + \cdots + q^{w-1}) = m_{BD} \pi_{BD} \frac{q^a - q^w}{1-q}.
\]

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} + m_{BD^{(2)}} \pi_{BD^{(2)}} =
\]
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W,w}[X_{k+1}; q] = \frac{x_{k-q}^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_{k-i} \rightarrow q x_{k-i}; 1 \leq i \leq w}
\]

Proof

\( P_2[x_{-1}, x_0, x_1; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1 \)

Suppose that \( m_{BD}[X_k] \pi_{BD}(q) \) is a summand of \( P_W[X_k; q] \)

Suppose that exactly \( a \) of the variables \( x_{i-1}, \ldots, x_{i-w} \) are in the monomial \( m_{BD}[X_k] \)

This given, we can obtain two of the properly colored bar diagrams of the schedule \( W, w \)

by appending to \( BD \) first an up bar of length \( w \) with \( w-a \) red upper cells

and then appending to \( BD \) a down bar of length \( w \) with \( a \) red lower cells.

Calling \( BD^{(1)} \) and \( BD^{(2)} \) the resulting colored bar diagrams of \( W, w \).

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} = x_k m_{BD} \pi_{BD} (1 + \cdots + q^{a-1}) = m_{BD} \pi_{BD} \frac{x_k - x_k q^a}{1-q}
\]

\[
m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD} (q^a + \cdots + q^{w-1}) = m_{BD} \pi_{BD} \frac{q^a - q^w}{1-q}
\]

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} + m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD} \frac{x_k - q^w}{1-q} + \frac{1-x_k}{1-q} m_{BD} \pi_{BD} \bigg|_{x_{k-i} \rightarrow q x_{k-i}; 1 \leq i \leq w}
\]
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem

For any schedule $W = (w_1, w_2, \ldots, w_{k-1})$ and any $1 \leq w \leq w_{k-1} + 1$ we have

$$P_{W,w}[X_{k+1};q] = \frac{x_k-q^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_{k-i}\to qx_{k-i}} ; 1 \leq i \leq w$$

Proof

$P_2[x_{-1}, x_0, x_1; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1$

Suppose that $m_{BD}[X_k] \pi_{BD}(q)$ is a summand of $P_W[X_k; q]$.

Suppose that exactly $a$ of the variables $x_{i-1}, \ldots, x_{i-w}$ are in the monomial $m_{BD}[X_k]$.

This given, we can obtain two of the properly colored bar diagrams of the schedule $W, w$ by appending to $BD$ first an up bar of length $w$ with $w - a$ red upper cells and then appending to $BD$ a down bar of length $w$ with $a$ red lower cells.

Calling $BD^{(1)}$ and $BD^{(2)}$ the resulting colored bar diagrams of $W, w$.

$$m_{BD^{(1)}} \pi_{BD^{(1)}} = x_k m_{BD} \pi_{BD}(1 + \cdots + q^{a-1}) = m_{BD} \pi_{BD} \frac{x_k - q^w}{1-q}$$

$$m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD}(q^a + \cdots + q^{w-1}) = m_{BD} \pi_{BD} \frac{q^a - q^w}{1-q}.$$

$$m_{BD^{(1)}} \pi_{BD^{(1)}} + m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD} \frac{x_k - q^w}{1-q} + \frac{1-x_k}{1-q} m_{BD} \pi_{BD} \bigg|_{x_{k-i}\to qx_{k-i}} ; 1 \leq i \leq w$$
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W,w}[X_{k+1}; q] = \frac{x_k - q^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_{k-i} \rightarrow qx_{k-i}} ; 1 \leq i \leq w
\]

Proof

\( P_2[x_{-1}, x_0, x_1]; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1 \)

Suppose that \( m_{BD}[X_k] \pi_{BD}(q) \) is a summand of \( P_W[X_k; q] \)

Suppose that exactly \( a \) of the variables \( x_{i-1}, \ldots, x_{i-w} \) are in the monomial \( m_{BD}[X_k] \).

This given, we can obtain two of the properly colored bar diagrams of the schedule \( W, w \)

by appending to \( BD \) first an up bar of length \( w \) with \( w - a \) red upper cells

and then appending to \( BD \) a down bar of length \( w \) with \( a \) red lower cells.

Calling \( BD^{(1)} \) and \( BD^{(2)} \) the resulting colored bar diagrams of \( W, w \).

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} = x_k m_{BD} \pi_{BD} (1 + \cdots + q^{a-1}) = m_{BD} \pi_{BD} \frac{x_k - x_k q^a}{1-q}
\]

\[
m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD} (q^a + \cdots + q^{w-1}) = m_{BD} \pi_{BD} \frac{q^a - q^w}{1-q}.
\]

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} + m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD} \frac{x_k - q^w}{1-q} + \frac{1-x_k}{1-q} m_{BD} \pi_{BD} \bigg|_{x_{k-i} \rightarrow qx_{k-i}} ; 1 \leq i \leq w
\]
Theorem

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W,w}[X_{k+1};q] = \frac{x_k-q^w}{1-q} P_W[X_k;q] + \frac{1-x_k}{1-q} P_W[X_k;q] \bigg|_{x_k \rightarrow qx_k} ; 1 \leq i \leq w
\]

Proof

\( P_2[x_{-1}, x_0, x_1]; q \) = \( q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1 \)

Suppose that \( m_{BD}[X_k] \pi_{BD}(q) \) is a summand of \( P_W[X_k;q] \)

Suppose that exactly \( a \) of the variables \( x_{i-1}, \ldots, x_{i-w} \) are in the monomial \( m_{BD}[X_k] \).

This given, we can obtain two of the properly colored bar diagrams of the schedule \( W, w \) by appending to \( BD \) first an up bar of length \( w \) with \( w-a \) red upper cells and then appending to \( BD \) a down bar of length \( w \) with \( a \) red lower cells.

Calling \( BD^{(1)} \) and \( BD^{(2)} \) the resulting colored bar diagrams of \( W, w \).

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} = x_k m_{BD} \pi_{BD}(1 + \cdots + q^{a-1}) = m_{BD} \pi_{BD} \frac{x_k-q^a}{1-q}
\]

\[
m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD}(q^a + \cdots + q^{w-1}) = m_{BD} \pi_{BD} \frac{q^a-q^w}{1-q}.
\]

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} + m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD} \frac{x_k-q^w}{1-q} + \frac{1-x_k}{1-q} m_{BD} \pi_{BD} \bigg|_{x_k \rightarrow qx_k} ; 1 \leq i \leq w
\]
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem

For any schedule $W = (w_1, w_2, \ldots, w_{k-1})$ and any $1 \leq w \leq w_{k-1} + 1$ we have

$$ P_{W,w}[X_{k+1}; q] = \frac{x_{k-w}q^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_{k-i} \rightarrow q x_{k-i}} \quad 1 \leq i \leq w $$

Proof

$P_2[x_{-1}, x_0, x_1; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1$

Suppose that $m_{BD}[X_k] \pi_{BD}(q)$ is a summand of $P_W[X_k; q]$. Suppose that exactly $a$ of the variables $x_{i-1}, \ldots, x_{i-w}$ are in the monomial $m_{BD}[X_k]$. This given, we can obtain two of the properly colored bar diagrams of the schedule $W, w$ by appending to $BD$ first an up bar of length $w$ with $w - a$ red upper cells and then appending to $BD$ a down bar of length $w$ with $a$ red lower cells.

Calling $BD^{(1)}$ and $BD^{(2)}$ the resulting colored bar diagrams of $W, w$.

$$ m_{BD^{(1)}} \pi_{BD^{(1)}} = x_k m_{BD} \pi_{BD}(1 + \cdots + q^{a-1}) = m_{BD} \pi_{BD} \frac{x_k - x_k q^a}{1-q} $$

$$ m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD}(q^a + \cdots + q^{w-1}) = m_{BD} \pi_{BD} \frac{q^a - q^w}{1-q} $$

$$ m_{BD^{(1)}} \pi_{BD^{(1)}} + m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD} \frac{x_k - q^w}{1-q} + \frac{1-x_k}{1-q} m_{BD} \pi_{BD} \bigg|_{x_{k-i} \rightarrow q x_{k-i}} \quad 1 \leq i \leq w $$
Theorem

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W_i, w}[X_{k+1}; q] = \frac{x_k - q^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_k-i \rightarrow qx_k-i} \; ; \; 1 \leq i \leq w
\]

Proof

\( P_2[x_{-1}, x_0, x_1]; q \) = \( q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1 \)

Suppose that \( m_{BD}[X_k] \pi_{BD}(q) \) is a summand of \( P_W[X_k; q] \)

Suppose that exactly \( a \) of the variables \( x_{i-1}, \ldots, x_{i-w} \) are in the monomial \( m_{BD}[X_k] \).

This given, we can obtain two of the properly colored bar diagrams of the schedule \( W, w \) by appending to \( BD \) first an up bar of length \( w \) with \( w - a \) red upper cells

and then appending to \( BD \) a down bar of length \( w \) with \( a \) red lower cells.

Calling \( BD^{(1)} \) and \( BD^{(2)} \) the resulting colored bar diagrams of \( W, w \).

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} = x_k m_{BD} \pi_{BD}(1 + \cdots + q^{a-1}) = m_{BD} \pi_{BD} \frac{x_k - q^a}{1-q}
\]

\[
m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD}(q^a + \cdots + q^{w-1}) = m_{BD} \pi_{BD} \frac{q^a - q^w}{1-q}.
\]

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} + m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD} \frac{x_k - q^w}{1-q} + \frac{1-x_k}{1-q} m_{BD} \pi_{BD} \bigg|_{x_k-i \rightarrow qx_k-i} \; ; \; 1 \leq i \leq w
\]
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W,w}[X_{k+1}; q] = \frac{x_k - q^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_{k-i} \rightarrow qx_{k-i}; 1 \leq i \leq w}
\]

Proof

\[P_2[x_{-1}, x_0, x_1]; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1\]

Suppose that \( m_{BD}[X_k] \pi_{BD}(q) \) is a summand of \( P_W[X_k; q] \).

Suppose that exactly \( a \) of the variables \( x_{i-1}, \ldots, x_{i-w} \) are in the monomial \( m_{BD}[X_k] \).

This given, we can obtain two of the properly colored bar diagrams of the schedule \( W, w \) by appending to \( BD \) first an up bar of length \( w \) with \( w - a \) red upper cells and then appending to \( BD \) a down bar of length \( w \) with \( a \) red lower cells.

Calling \( BD^{(1)} \) and \( BD^{(2)} \) the resulting colored bar diagrams of \( W, w \).

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} = x_k m_{BD} \pi_{BD}(1 + \cdots + q^{a-1}) = m_{BD} \pi_{BD} \frac{x_k - q^a}{1-q}
\]

\[
m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD}(q^a + \cdots + q^{w-1}) = m_{BD} \pi_{BD} \frac{q^a - q^w}{1-q}.
\]

\[
m_{BD^{(1)}} \pi_{BD^{(1)}} + m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD} \frac{x_k - q^w}{1-q} + \frac{1-x_k}{1-q} m_{BD} \pi_{BD} \bigg|_{x_{k-i} \rightarrow qx_{k-i}; 1 \leq i \leq w}
\]
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W,w}[X_{k+1}; q] = \frac{x_k-q^w}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_{k-i} \to q x_{k-i}; 1 \leq i \leq w}
\]

Proof

\[
\begin{align*}
P_2[x_{-1}, x_0, x_1; q] &= q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1 \\

\text{Suppose that } m_{BD}[X_k] \pi_{BD}(q) \text{ is a summand of } P_W[X_k; q] \\

\text{Suppose that exactly } a \text{ of the variables } x_{i-1}, \ldots, x_{i-w} \text{ are in the monomial } m_{BD}[X_k]. \\

\text{This given, we can obtain two of the properly colored bar diagrams of the schedule } W, w \\
\text{by appending to } BD \text{ first an up bar of length } w \text{ with } w - a \text{ red upper cells} \\
\text{and then appending to } BD \text{ a down bar of length } w \text{ with } a \text{ red lower cells.} \\

\text{Calling } BD^{(1)} \text{ and } BD^{(2)} \text{ the resulting colored bar diagrams of } W, w. \\

m_{BD^{(1)}} \pi_{BD^{(1)}} = x_k m_{BD} \pi_{BD}(1 + \cdots + q^{a-1}) = m_{BD} \pi_{BD} \frac{x_k - q^a}{1-q} \\
m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD}(q^a + \cdots + q^{w-1}) = m_{BD} \pi_{BD} \frac{q^a - q^w}{1-q}. \\
m_{BD^{(1)}} \pi_{BD^{(1)}} + m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD} \frac{x_k - q^w}{1-q} + \frac{1-x_k}{1-q} m_{BD} \pi_{BD} \bigg|_{x_{k-i} \to q x_{k-i}; 1 \leq i \leq w}
\end{align*}
\]
CONSTRUCTION OF THIS REMARKABLE POLYNOMIAL

Theorem

For any schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we have

\[
P_{W,w}[X_{k+1}; q] = \frac{x_{k-q^w}}{1-q} P_W[X_k; q] + \frac{1-x_k}{1-q} P_W[X_k; q] \bigg|_{x_{k-i} \rightarrow qx_{k-i}; 1 \leq i \leq w}
\]

Proof

\[P_2[x_{-1}, x_0, x_1]; q] = q^2 x_{-1} + q x_0 + q x_{-1} x_1 + x_0 x_1\]

Suppose that \( m_{BD}[X_k] \pi_{BD}(q) \) is a summand of \( P_W[X_k; q] \).

Suppose that exactly \( a \) of the variables \( x_{i-1}, \ldots, x_{i-w} \) are in the monomial \( m_{BD}[X_k] \).

This given, we can obtain two of the properly colored bar diagrams of the schedule \( W, w \) by appending to \( BD \) first an up bar of length \( w \) with \( w - a \) red upper cells

and then appending to \( BD \) a down bar of length \( w \) with \( a \) red lower cells.

Calling \( BD^{(1)} \) and \( BD^{(2)} \) the resulting colored bar diagrams of \( W, w \).

\[m_{BD^{(1)}} \pi_{BD^{(1)}} = x_k m_{BD} \pi_{BD}(1 + \cdots + q^{a-1}) = m_{BD} \pi_{BD} \frac{x_k-q^a}{1-q}\]

\[m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD}(q^a + \cdots + q^{w-1}) = m_{BD} \pi_{BD} \frac{q^a-q^w}{1-q}\]

\[m_{BD^{(1)}} \pi_{BD^{(1)}} + m_{BD^{(2)}} \pi_{BD^{(2)}} = m_{BD} \pi_{BD} \frac{x_k-q^w}{1-q} + \frac{1-x_k}{1-q} m_{BD} \pi_{BD} \bigg|_{x_{k-i} \rightarrow qx_{k-i}; 1 \leq i \leq w}\]
THE FUNCTIONAL EQUATION
THE FUNCTIONAL EQUATION

Define
Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]
Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]
Define
\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]

Conjecture I

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion
Define

$$Q_W(x; q) = P_W[X_k; q]_{x_i = x; \forall i}$$

**Conjecture I**

For all legal schedules $W = (w_1, w_2, \ldots, w_{k-1}$, the coefficients $A_s(q)$ in the expansion

$$Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s$$
THE FUNCTIONAL EQUATION

Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]

Conjecture I

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities
Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]

**Conjecture I**

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities

\[ A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad (\text{for all } 1 \leq s \leq (k+1)/2) \]
Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]

**Conjecture I**

*For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}, \) the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities

\[ A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad (\text{for all } 1 \leq s \leq (k + 1)/2) \]

where the polynomials on both sides have non-negative integer coefficients*
Define

\[
Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i}
\]

Conjecture I

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[
Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s
\]

satisfy the identities

\[
A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad \text{(for all } 1 \leq s \leq (k + 1)/2)\]

where the polynomials on both sides have non-negative integer coefficients

Theorem
Define

\[ Q_W(x; q) = P_W[X_k; q] \mid_{x_i = x; \forall i} \]

Conjecture I

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities

\[ A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad (\text{for all } 1 \leq s \leq (k + 1)/2) \]

where the polynomials on both sides have non-negative integer coefficients

Theorem

Conjecture I minus the positivity assertion holds true if and only if
Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i=x; \forall i} \]

**Conjecture I**

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities

\[ A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad \text{(for all } 1 \leq s \leq (k + 1)/2) \]

where the polynomials on both sides have non-negative integer coefficients.

**Theorem**

Conjecture I minus the positivity assertion holds true if and only if

for any legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) we have
THE FUNCTIONAL EQUATION

Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]

Conjecture I

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities

\[ A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad (\text{for all } 1 \leq s \leq (k+1)/2) \]

where the polynomials on both sides have non-negative integer coefficients

Theorem

Conjecture I minus the positivity assertion holds true if and only if

for any legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \), we have

\[ (1 - \frac{q}{x})Q_W(x; q) + x^k(1 - qx)Q_W(1/x; q) = (1 + x^k)(A_k(q) - qA_1(q)) \]
Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]

Conjecture I

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities

\[ A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad (\text{for all } 1 \leq s \leq (k + 1)/2) \]

where the polynomials on both sides have non-negative integer coefficients.

Theorem

Conjecture I minus the positivity assertion holds true if and only if

for any legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \), we have

\[ (1 - \frac{q}{x})Q_W(x; q) + x^k(1 - qx)Q_W(1/x; q) = (1 + x^k)(A_k(q) - qA_1(q)) \]

In particular it follows that
THE FUNCTIONAL EQUATION

Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]

**Conjecture I**

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities

\[ A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad (\text{for all } 1 \leq s \leq (k + 1)/2) \]

where the polynomials on both sides have non-negative integer coefficients.

**Theorem**

Conjecture I minus the positivity assertion holds true if and only if for any legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) we have

\[ (1 - \frac{q}{x})Q_W(x; q) + x^k(1 - qx)Q_W(1/x; q) = (1 + x^k)(A_k(q) - qA_1(q)) \]

In particular it follows that

\[ A_k(q) - qA_1(q) = (1 - q^2) \prod_{i=1}^{k-1} [w_i]_q \]
Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]

**Conjecture I**

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities

\[ A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad \text{(for all } 1 \leq s \leq (k+1)/2 \text{)} \]

where the polynomials on both sides have non-negative integer coefficients.

**Theorem**

Conjecture I minus the positivity assertion holds true if and only if for any legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \), we have

\[ (1 - \frac{q}{x})Q_W(x; q) + x^k (1 - qx)Q_W(1/x; q) = (1 + x^k)(A_k(q) - qA_1(q)) \]

In particular it follows that

\[ A_k(q) - qA_1(q) = (1 - q^2) \prod_{i=1}^{k-1} [w_i]_q \]
THE FUNCTIONAL EQUATION

Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]

Conjecture I

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities

\[ A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad \text{(for all } 1 \leq s \leq (k+1)/2) \]

where the polynomials on both sides have non-negative integer coefficients.

Theorem

Conjecture I minus the positivity assertion holds true if and only if

for any legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) we have

\[ (1 - \frac{q}{x})Q_W(x; q) + x^k(1 - qx)Q_W(1/x; q) = (1 + x^k)(A_k(q) - qA_1(q)) \]

In particular it follows that

\[ A_k(q) - qA_1(q) = (1 - q^2) \prod_{i=1}^{k-1} [w_i]_q \]
THE FUNCTIONAL EQUATION

Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]

Conjecture I

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities

\[ A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad \text{(for all } 1 \leq s \leq (k + 1)/2) \]

where the polynomials on both sides have non-negative integer coefficients

Theorem

Conjecture I minus the positivity assertion holds true if and only if

for any legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \), we have

\[ (1 - \frac{q}{x})Q_W(x; q) + x^k(1 - qx)Q_W(1/x; q) = (1 + x^k)(A_k(q) - qA_1(q)) \]

In particular it follows that

\[ A_k(q) - qA_1(q) = (1 - q^2) \prod_{i=1}^{k-1} [w_i]_q \]
Define

\[
Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i=x; \forall i}
\]

Conjecture I

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[
Q_W(x; q) = \sum_{s=1}^k A_s(q)x^s
\]

satisfy the identities

\[
A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad (\text{for all } 1 \leq s \leq (k+1)/2)
\]

where the polynomials on both sides have non-negative integer coefficients.

Theorem

Conjecture I minus the positivity assertion holds true if and only if

for any legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) we have

\[
(1 - \frac{q}{x})Q_W(x; q) + x^k(1 - qx)Q_W(1/x; q) = (1 + x^k)(A_k(q) - qA_1(q))
\]

In particular it follows that

\[
A_k(q) - qA_1(q) = (1 - q^2) \prod_{i=1}^{k-1} [w_i]_q
\]
Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i = x; \forall i} \]

**Conjecture I**

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities

\[ A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad (\text{for all } 1 \leq s \leq (k + 1)/2) \]

where the polynomials on both sides have non-negative integer coefficients

**Theorem**

Conjecture I minus the positivity assertion holds true if and only if for any legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) we have

\[ (1 - \frac{q}{x})Q_W(x; q) + x^k(1 - qx)Q_W(1/x; q) = (1 + x^k)(A_k(q) - qA_1(q)) \]

In particular, it follows that

\[ A_k(q) - qA_1(q) = (1 - q^2) \prod_{i=1}^{k-1} [w_i]_q \]
Define

\[ Q_W(x; q) = P_W[X_k; q] \bigg|_{x_i=x; \forall i} \]

**Conjecture I**

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion

\[ Q_W(x; q) = \sum_{s=1}^{k} A_s(q)x^s \]

satisfy the identities

\[ A_s(q) - qA_{s+1}(q) = qA_{k+1-s}(q) - A_{k-s}(q) \quad (\text{for all } 1 \leq s \leq (k + 1)/2) \]

where the polynomials on both sides have non-negative integer coefficients

**Theorem**

Conjecture I minus the positivity assertion holds true if and only if

for any legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) we have

\[ (1 - \frac{q}{x})Q_W(x; q) + x^k(1 - qx)Q_W(1/x; q) = (1 + x^k)(A_k(q) - qA_1(q)) \]

In particular it follows that

\[ A_k(q) - qA_1(q) = (1 - q^2) \prod_{i=1}^{k-1} [w_i]_q \]
OUR GREAT EVENT OF 2017
OUR GREAT EVENT OF 2017

Theorem
OUR GREAT EVENT OF 2017

Theorem

For all legal schedules $W$ the polynomials $Q_W[x; q)$
Theorem

For all legal schedules $W$ the polynomials $Q_W[x; q]$ satisfy the Functional equation.
OUR GREAT EVENT OF 2017

Theorem

For all legal schedules $W$ the polynomials $Q_W[x; q)$ satisfy the Functional equation

This reduces the compositional Shuffle Conjecture
Theorem

For all legal schedules \( W \) the polynomials \( Q_W[x; q] \) satisfy the Functional equation

This reduces the compositional Shuffle Conjecture to the partition case, where both sides give
THEOREM

For all legal schedules $W$ the polynomials $Q_W[x; q)$ satisfy the Functional equation

This reduces the compositional Shuffle Conjecture to the partition case, where both sides give a symmetric function basis.
OUR GREAT EVENT OF 2017

Theorem

For all legal schedules $W$ the polynomials $Q_W[x; q)$ satisfy the Functional equation

This reduces the compositional Shuffle Conjecture to the partition case, where both sides give a symmetric function basis.

THANKYOU
OUR GREAT EVENT OF 2017

Theorem

For all legal schedules $W$ the polynomials $Q_W[x; q)$ satisfy the Functional equation

This reduces the compositional Shuffle Conjecture
to the partition case, where both sides give

a symmetric function basis.

THANKYOU
Theorem

For all legal schedules $W$ the polynomials $Q_W[x; q]$ satisfy the Functional equation

This reduces the compositional Shuffle Conjecture to the partition case, where both sides give a symmetric function basis.

THANKYOU
Theorem

For all legal schedules $W$ the polynomials $Q_W[x; q]$ satisfy the Functional equation

This reduces the compositional Shuffle Conjecture to the partition case, where both sides give a symmetric function basis.

THANKYOU
OUR GREAT EVENT OF 2017

Theorem

For all legal schedules $W$ the polynomials $Q_W[x; q)$ satisfy the Functional equation

This reduces the compositional Shuffle Conjecture to the partition case, where both sides give a symmetric function basis.

THANKYOU
Theorem

For all legal schedules $W$ the polynomials $Q_W[x; q]$ satisfy the Functional equation

This reduces the compositional Shuffle Conjecture to the partition case, where both sides give a symmetric function basis.

THANKYOU
Theorem

For all legal schedules $W$ the polynomials $Q_W[x; q]$ satisfy the Functional equation

This reduces the compositional Shuffle Conjecture to the partition case, where both sides give a symmetric function basis.

THANKYOU