Representations on real toric spaces

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*Geometric representations of finite groups on real toric spaces*

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Goal

To understand the Weyl group representation on the (co)homology of real toric variety associated to Weyl chambers.
Toric variety

A toric variety is a normal variety $X$ that contains a torus $T = (\mathbb{C}^*)^n$ as a dense open subset together with an action $T \times X \to X$ of $T$ on $X$ that extends the natural action of $T$ on itself.

$\text{toric varieties} \iff \text{fans}$

$\mathbb{C}P^2 \iff \left( K = \{\emptyset, 1, 2, 3, 12, 13, 23\}, \Lambda = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \right)$
Real toric variety

For a toric variety $X$, real toric variety $X^\mathbb{R}$ of $X$ is the real locus of $X$; that is, $X^\mathbb{R}$ is the set of points that are stable under the canonical involution.

Example

$$(\mathbb{C}P^2)^\mathbb{R} = \mathbb{R}P^2$$

For $X = (K, \Lambda)$, if $K$ a simplicial complex on $[m]$ let

$$\mathbb{R}\mathcal{Z}_K = \bigcup_{\sigma \in K} \{ (x_1, \ldots, x_m) \in (D^1)^m \mid x_i \in S^0 \text{ when } i \notin \sigma \}$$

where

$$D^1 = [-1, 1], \quad S^0 = \{-1, 1\},$$

and $\Lambda_2$ be the 0, 1 matrix obtained from $\Lambda$ by taking mod 2 values. Then

$$X^\mathbb{R} = \mathbb{R}\mathcal{Z}_K / \ker \Lambda_2$$
Toric variety associated to Weyl chambers [1990 Procesi]

\[ \Phi: \text{(irreducible) root system of type } R \text{ and rank } n \]

\[ W_R: \text{Weyl group of type } R \]

\[ K_R: \text{Coxeter complex of type } R \]

\[ R = A_2 \]

\[ K_{A_2} = \{ \emptyset, 1, 2, 3, 4, 5, 6, 12, 23, 34, 45, 56, 16 \} \]

\[ \Lambda = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix} \]

\[ \Lambda_2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \]
(Co)homology

\[ X = (K, \Lambda) \text{ with } K \text{ a simplicial complex on } [m] \]
\[ X^R : \text{real toric variety of } X \]

Theorem

\[ H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_m]/(I + J) \]

Theorem [Suciu 2012, Choi-Park 2017]

\[ H^*(X^R; \mathbb{Q}) \cong \bigoplus_{\omega \in \text{row} \Lambda_2} \tilde{H}^{*-1}(K_\omega; \mathbb{Q}) \]

\[ H_*(X^R; \mathbb{Q}) \cong \bigoplus_{\omega \in \text{row} \Lambda_2} \tilde{H}_{*-1}(K_\omega; \mathbb{Q}) \]

as \( \mathbb{Q} \) vector spaces.
Weyl group action on the cohomology of $X_R$

$\mathcal{W}$-action on $H^*(X_R; \mathbb{Q})$ induced from the action on the Coxeter complex

1990 Procesi
1992, 1994 Stembridge

Theorem [Betti number]

$$\dim(H^{2k}(X_{A_n}; \mathbb{Q})) = A(n, k + 1)$$
Weyl group action on homology of $X^R_R$

$W$-action on $H_\ast(X^R_R; \mathbb{Q})$ induced from the action on the Coxeter complex

Remark
Since irreducible representations of Weyl groups are self-dual and the cohomology and the homology of $X^R_R$ are dual representations, they are isomorphic representations.

Theorem [Suciu 2012, Choi-Park 2017]

$$H_\ast(X^R_R; \mathbb{Q}) \cong \bigoplus_{\omega \in \text{row}\Lambda_2} \tilde{H}_{\ast-1}((K_R)_\omega; \mathbb{Q}) \quad (\ast)$$

as $\mathbb{Q}$ vector spaces.
Theorem [C-Choi-Kaji, 2017]

Let $X = (K, \Lambda)$ and suppose a finite group $G$ acts on the simplicial complex $K$. Then, the following are equivalent:

1. $G$ preserves $\ker \Lambda_2$;
2. the action induces one on $X^R = \mathbb{R} \mathbb{Z}_K / \ker \Lambda_2$;
3. each element of $G$ permutes columns of $\Lambda_2$ without changing its row space;
4. each element $g \in G$ permutes columns of $\Lambda_2$ in such a way that there exists an $n \times n$-matrix $A_g$ such that $g \Lambda_2 = A_g \Lambda_2$;
5. there exists a $G$-action on $H_*(X^R)$ which is compatible with the isomorphism $(\ast)$, where the action of $g \in G$ on the right hand side is induced by $g : K_\omega \rightarrow K_{g\omega}$. 

Weyl group action on homology of $X^R$
Weyl group action on homology of $X^R_R$

Theorem [C-Choi-Kaji 2017]

The Weyl group $W_R$ acts on $K_R$ and preserves $\ker \Lambda_2$. More precisely, let $\Lambda^j_2 \in \mathbb{Z}_2^m$ be the $j$th row of $\Lambda_2$, which corresponds to the $\omega_j$ coordinates of the rays. Then, we have

$$(s_i(\Lambda_2))^j = \Lambda^j_2 - c_{ij} \Lambda_2 R^i,$$

where $c_{ij} = (\alpha_i^\vee, \alpha_j)$ are the entries of the Cartan matrix of $R$. Hence we have

$$H_*(X^R_R; \mathbb{Q}) \cong \bigoplus_{\omega \in \text{row} \Lambda_2} \tilde{H}_{*-1}((K_R)_\omega; \mathbb{Q}) \quad (\ast)$$

as $W_R$ modules.
Type A representation

\[ W_{A_n} = S_{n+1}. \]
\[ \omega \in Row((\Lambda_{A_n})_2) \iff S_\omega \subset [n] \]

Theorem [Choi-Park, 2015]

For \( S \subset [n] \), let \( I_S = S \) if \( |S| \) is even and \( I_S = S \cup \{n + 1\} \) otherwise. Then \( (K_{A_n})_\omega \) is homotopically equivalent to the odd rank-selected Boolean algebra \( B_{I_{S_\omega}}^{odd} \).

Theorem [Solomon 1968, Stanley 1982]

Let \( Q \subset [m - 1] \). Then the homology of the \( Q \)-rank-selected poset \( B_{[m]}^Q \) is given, as an \( S_m \)-module, by

\[
\tilde{H}_*(B_{[m]}^Q) \cong \begin{cases} 
\bigoplus_\nu c_{Q,\nu} S_\nu & (* = |Q| - 1) \\
0 & (* \neq |Q| - 1),
\end{cases}
\]

where \( c_{Q,\nu} \) is the number of standard tableaux of shape \( \nu \) with descent set \( Q \).
Type A representation

Theorem [C-Choi-Kaji 2017]

Let $Q = \{1, 3, \ldots, 2r - 1\}$ and $c_{Q,\nu}$ be the number of standard tableaux of shape $\nu$ with descent set $Q$. Then, we have

$$H_r(X_{A_n}^\mathbb{R}) \cong \bigoplus_{\eta \vdash (n+1)} \left( \sum_{\nu} c_{Q,\nu} \right) S^\eta,$$

where $\nu$ runs over all partitions of $2r$ that is contained in $\eta$, and $\eta/\nu$ has at most one box in each column.

Proof

$$H_r(X_{A_n}^\mathbb{R}) \cong \text{Ind}_{\mathfrak{S}_{\{1,\ldots,2r\}} \times \mathfrak{S}_{\{2r+1,\ldots,n+1\}}}^{\mathfrak{S}_{n+1}} \left( \bigoplus_{\nu} c_{Q,\nu} S^\nu \otimes S^{(n-2r)} \right)$$

$$\cong \bigoplus_{\nu} c_{Q,\nu} \left( \text{Ind}_{\mathfrak{S}_{\{1,\ldots,2r\}} \times \mathfrak{S}_{\{2r+1,\ldots,n+1\}}}^{\mathfrak{S}_{n+1}} \left( S^\nu \otimes S^{(n-2r)} \right) \right)$$

$$\cong \bigoplus_{\nu} c_{Q,\nu} \left( \bigoplus_{\nu \rightarrow \eta} S^\eta \right) \cong \bigoplus_{\eta \vdash (n+1)} \left( \sum_{\nu \rightarrow \eta} c_{Q,\nu} \right) S^\eta.$$
Type A representation

Let $a_n$ be the number of alternating permutations (snakes) in $S_{n+1}$.

**Theorem [Hendersen 2012]**

The $r$th Betti number of $X_{A_n}^{\mathbb{R}}$ is $\binom{n+1}{2r} a_{2r}$.

**Corollary [C-Choi-Kaji 2017]**

The $r$th Betti number of $X_{A_n}^{\mathbb{R}}$, which is known to be $\binom{n+1}{2r} a_{2r}$, is

$$\sum_{\eta \vdash (n+1)} \left( \sum_{\nu} c_{Q,\nu} \right) f^{\eta}.$$

**Example**

$$H_3(X_{A_5}^{\mathbb{R}}) \cong S^{(3,3)} \oplus 2S^{(3,2,1)} \oplus S^{(3,1,1,1)} \oplus S^{(2,2,2)} \oplus S^{(2,2,1,1)}.$$
Type $B$ representation

$W_{B_n}$ is the group of signed permutations on $[n]$.
$\omega \in \text{Row}((\Lambda_{B_n})_2) \iff S_\omega \subset [n]$

Theorem [Choi-Park-Park, 2017]

$(K_{B_n})_\omega$ is homotopically equivalent to the odd rank-selected lattice $C_{S_\omega}^{odd}$ of faces of the cross-polytope over $S_\omega$.

Theorem [Stanley 1982]

When $|S_\omega| = r$,

$$\widetilde{H}_*(C_{S_\omega}^{odd}) \cong \begin{cases} \bigoplus_{(\lambda,\mu)\vdash r} b(\lambda,\mu) S^{(\lambda,\mu)} & (\ast = \left\lfloor \frac{r-1}{2} \right\rfloor) \\ 0 & (\ast \neq \left\lfloor \frac{r-1}{2} \right\rfloor) \end{cases},$$

where $b(\lambda,\mu)$ is the number of double standard Young tableaux of shape $(\lambda,\mu)$ whose descent set is the set of odd numbers less than or equal to $r = |\lambda| + |\mu|$.
**Type B representation**

**Theorem [C-Choi-Kaji 2017]**

The $k$th homology $H_k(X_{B_n}^\mathbb{R})$ of $X_{B_n}^\mathbb{R}$ with the natural action of $W_{B_n}$ is isomorphic to the sum of two induced representations

$$
\bigoplus_{r \in \{2k-1, 2k\}} \left( \text{Ind}_{W_{Br} \times W_{B_{n-r}}}^{W_{B_n}} \left( \bigoplus_{(\lambda, \mu) \vdash r} b(\lambda, \mu) S^{(\lambda, \mu)} \otimes S^{(\emptyset; (n-r))} \right) \right)
$$

of $W_{B_n}$, where $S^{(\emptyset; (n-r))}$ is the trivial representation of $W_{B_{n-r}}$.

**Corollary [C-Choi-Kaji 2017]**

$$
H_k(X_{B_n}^\mathbb{R}) \cong \bigoplus_{(\lambda, \nu) \vdash n} \left( \sum_{r \in \{2k-1, 2k\}} \sum_{(\lambda, \mu) \vdash r, \mu \sim \nu} b(\lambda, \mu) \right) S^{(\lambda, \nu)},
$$

where $\mu$ in the inside summation, runs over all partitions that is contained in $\nu$, and $\nu/\mu$ has $(n - r)$ boxes with at most one box in each column.
**Type B representation**

Let $b_n$ be the number of alternating signed permutations (snakes) in $W_{B_n}$.

**Theorem [Choi-Park-Park 2017]**

The $r$th Betti number of $X_{B_n}^\mathbb{R}$ is $\binom{n}{2r} b_{2r} + \binom{n}{2r-1} b_{2r-1}$.

**Corollary [C-Choi-Kaji 2017]**

The $r$th Betti number of $X_{A_n}^\mathbb{R}$, which is known to be $\binom{n}{2r} b_{2r} + \binom{n}{2r-1} b_{2r-1}$, is

$$\sum_{(\lambda,\nu) \vdash n} \left( \sum_{r \in \{2k-1, 2k\}} \sum_{(\lambda,\mu) \vdash r, \mu \sim \nu} b(\lambda, \mu) \right) \binom{n}{|\lambda|} f^\lambda f^\nu.$$

**Example**

$$H_2(X_{B_3}^\mathbb{R}) \cong S^{((1),(1,1))} \oplus S^{((2),(1))} \oplus S^{((1,1),(1))} \oplus S^{((2),(1))}.$$

$$\left(\begin{array}{c} 1 \\ 3 \\ 2 \end{array}\right)$$ is the double standard tableau of shape $((1,1),(1))$ with descent set $\{1,3\}$. 
Thank You!
Thank You!

Congratulations!!