Physical Combinatorics

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Motivation

1988  Identity for Kostka polynomials Kerov, Kirillov, Reshetikhin
2001  $X = M$ conjecture of HKOTTY
Outline

1. Rogers-Ramanujan identities, fractional statistics, and the $X = M$ conjecture
2. Kirillov-Reshetikhin crystals
Rogers-Ramanujan identities

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1 - q^{5j+1})(1 - q^{5j+4})}
\]

\[
\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1 - q^{5j+2})(1 - q^{5j+3})}
\]

where \((q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)\) for \(n > 0\) and \((q)_0 = 1\).
Some History

• proved in a paper by Rogers in 1894
• conjectured by Ramanujan in a letter to Hardy in 1913; published in 1916 in the book *Combinatory Analysis* by MacMahon without proof
• new proof in 1917 by Rogers and Ramanujan
• different independent proof by Schur in 1917
Some History

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published in 1916 in the book *Combinatory Analysis* by MacMahon without proof
• new proof in 1917 by Rogers and Ramanujan
• different independent proof by Schur in 1917

Rogers-Schur-Ramanujan identities
Partition interpretation

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1 - q^{5j + 1})(1 - q^{5j + 4})}
\]

\[S = \{s_1, s_2, s_3, \ldots\} \text{ set}\]

\[
\prod_{n \in S} \frac{1}{1 - q^n} = \prod_{n \in S} (1 + q^n + q^{2n} + q^{3n} + \cdots)
\]

\[= (1 + q^{s_1} + q^{2s_1} + q^{3s_1} + \cdots) \times (1 + q^{s_2} + q^{2s_2} + q^{3s_2} + \cdots) \cdots .\]
Partition interpretation

\[ \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1 - q^{5j+1})(1 - q^{5j+4})} \]

\[ S = \{s_1, s_2, s_3, \ldots\} \text{ set} \]

\[ \prod_{n \in S} \frac{1}{1 - q^n} = \prod_{n \in S} (1 + q^n + q^{2n} + q^{3n} + \cdots) \]

\[ = (1 + q^{s_1} + q^{2s_1} + q^{3s_1} + \cdots) \times (1 + q^{s_2} + q^{2s_2} + q^{3s_2} + \cdots) \cdots. \]

**Theorem.** The product side is the generating function of partitions with parts congruent 1 or 4 modulo 5.
**Example:** The coefficient of $q^6$ is 3 since there are three partitions of 6 with parts congruent to 1 or 4 modulo 5:

$$(1, 1, 1, 1, 1, 1), \quad (4, 1, 1) \quad \text{and} \quad (6).$$
Example: The coefficient of $q^6$ is 3 since there are three partitions of 6 with parts congruent to 1 or 4 modulo 5:

$$(1, 1, 1, 1, 1), \quad (4, 1, 1) \quad \text{and} \quad (6).$$

Is there an interpretation of the sum side of the RR identities?
Some more history

- in 1990’s the Stony Brook group interpreted the Rogers–Ramanujan identities as the partition function of a physical system with quasiparticles obeying certain exclusion statistics ⇒ fermionic formulas
- HKOTTY in 1999/2001 conjectured fermionic formulas for all Kac–Moody Lie algebras ⇒ $X = M$ conjecture
The Hard Hexagon model

Set of paths:
height variable $\sigma_i \in \{0, 1\}$ for $0 \leq i \leq L$
boundary condition $\sigma_0 = \sigma_L = 0$
requirement $\sigma_i \sigma_{i+1} = 0$
The Hard Hexagon model

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requirement $\sigma_i \sigma_{i+1} = 0$

Example: Path of length 9

![Path of length 9 diagram]
Energy function

\[ E(p) = \sum_{j=1}^{L} j \sigma_j \]
\[ E(p) = 1 + 5 + 8 = 14 \]

Energy function

\[ E(p) = \sum_{j=1}^{L} j\sigma_j \]
Energy function

\[ E(p) = \sum_{j=1}^{L} j \sigma_j \]

Generating function

\[ X(L) = \sum_{p \text{ path of length } L} q^{E(p)} \]
Explicit formula

**Recurrence:** $X(L)$ is completely determined by $X(0) = X(1) = 1$ and

$$X(L) = X(L - 1) + q^{L-1}X(L - 2).$$

**Theorem.** $X(L) = \sum_{n=0}^{\infty} q^n \frac{L-n}{n} =: M(L)$

**Corollary.** $\lim_{L \to \infty} M(L) = \sum_{n=0}^{\infty} \frac{q^n}{(q)_n}$
Explicit formula

**Recurrence:** $X(L)$ is completely determined by $X(0) = X(1) = 1$ and

$$X(L) = X(L - 1) + q^{L-1}X(L - 2).$$

**Theorem.** $X(L) = \sum_{n=0}^{\infty} q^{n^2} \binom{L-n}{n} =: M(L)$

**Corollary.** $\lim_{L \to \infty} M(L) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}$

Sum side of the RR identities
Partition interpretation

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1 - q^{5j+1})(1 - q^{5j+4})}
\]

**Theorem.** The sum side is the generating function of partitions for which the difference between any two parts is at least two.
Partition interpretation

\[ \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1 - q^{5j+1})(1 - q^{5j+4})} \]

**Theorem.** The sum side is the generating function of partitions for which the difference between any two parts is at least two.

**Example.** Partitions of 6 with the difference between any two parts at least two are

\[ (4, 2), \quad (5, 1) \quad \text{and} \quad (6). \]
Statistics

**Bosons:** adding a particle does not remove any states from the system

\[
\sum_{m=0}^{\infty} \frac{q^m}{(q)_m} = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}
\]

**Fermions:** adding a particle removes one state from the system

\[
\sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} = \prod_{n=0}^{\infty} (1 + q^n)
\]
Fractional statistics

**RR identity:** interpret each triangle in a path as a particle; adding a particle removes two states from the system

\[ \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1 - q^{5j+1})(1 - q^{5j+4})} \]
Fractional statistics

**RR identity:** interpret each triangle in a path as a particle; adding a particle removes two states from the system

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1 - q^{5j+1})(1 - q^{5j+4})}
\]

**Haldane statistics:**
- \(d_a\): dimension of Hilbert space for particles of type \(a\)
- \(N_a\): number of particles of type \(a\)
- \(g_{ab}\): statistics matrix

\[
\Delta d_a = -\sum_b g_{ab} \Delta N_b
\]
Marriage

Citation from Dyson’s famous paper “Missed opportunities” (1972)

“As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce... I shall examine in detail some examples of missed opportunities, occasions on which mathematicians and physicists lost chances of making discoveries by neglecting to talk to each other.”
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\[ \leadsto \text{Kirillov–Reshetikhin (KR) crystals} \]
References

This talk is based on the following papers:

- A. Schilling, *Combinatorial structure of Kirillov–Reshetikhin crystals of type $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$*, preprint math.QA/0704.2046
Outline

Combinatorial structure of KR crystals of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$

- Crystals
- KR crystals
- Dynkin diagram automorphisms
- Classical crystal structure
- Affine crystal structure
- MuPAD-Combinat implementation
- Outlook and open problems
Quantum algebras

Drinfeld and Jimbo $\sim$ 1984:
independently introduced quantum groups $U_q(\mathfrak{g})$

Kashiwara $\sim$ 1990:
crystal bases, bases for $U_q(\mathfrak{g})$-modules as $q \to 0$
combinatorial approach

Lusztig $\sim$ 1990:
canonical bases
geometric approach
Applications in...

representation theory
\leadsto tensor product decomposition
solvable lattice models
\leadsto one point functions
conformal field theory
\leadsto characters
number theory
\leadsto modular forms
Bethe Ansatz
\leadsto fermionic formulas
combinatorics
\leadsto tableaux combinatorics
topological invariant theory
\leadsto knots and links
Crystals

g symmetrizable Kac-Moody algebra

$P$ weight lattice of $g$

$I$ index of the Dynkin diagram

$\{\alpha_i \in P \mid i \in I\}$ simple roots

$\{h_i \in P^* = \text{Hom}_\mathbb{Z}(P, \mathbb{Z}) \mid i \in I\}$ simple coroots
Crystals

A $U_q(g)$-crystal is a nonempty set $B$ with maps

$$\text{wt}: B \rightarrow P$$

$$e_i, f_i: B \rightarrow B \cup \{\emptyset\} \quad \text{for all } i \in I$$

satisfying

$$f_i(b) = b' \Leftrightarrow e_i(b') = b \quad \text{if } b, b' \in B$$

$$\text{wt}(f_i(b)) = \text{wt}(b) - \alpha_i \quad \text{if } f_i(b) \in B$$

$$\langle h_i, \text{wt}(b) \rangle = \varphi_i(b) - \epsilon_i(b)$$

Write $b \underset{i}{\rightarrow} b'$ for $b' = f_i(b)$
KR crystals

\( \mathfrak{g} \) affine Kac–Moody algebra

\( W^{r,s} \) KR module indexed by \( r \in \{1, \ldots, n\}, s \geq 1 \)

\( \sim \) finite-dimensional \( U'_q(\mathfrak{g}) \)-module

Chari proved

\[
W^{r,s} \cong \bigoplus_{\Lambda} W(\Lambda) \quad \text{as } U_q(\mathfrak{g}_0) \text{-module}
\]
KR crystals

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W^{r,s} \cong \bigoplus_{\Lambda} W(\Lambda) \quad \text{as } U_q(\mathfrak{g}_0) \text{-module}
\]

\( \mathfrak{g} \) of type \( A^{(1)}_n \implies \mathfrak{g}_0 \) of type \( A_n \)

\[
W^{r,s} \cong W \begin{pmatrix} \{ \} \end{pmatrix}^r_s
\]
KR crystals

$\mathfrak{g}$ affine Kac–Moody algebra

$W_{r,s}$ KR module indexed by $r \in \{1, \ldots, n\}$, $s \geq 1$

$\sim$ finite-dimensional $U'_q(\mathfrak{g})$-module

Chari proved

$$W_{r,s} \cong \bigoplus_{\Lambda} W(\Lambda) \quad \text{as } U_q(\mathfrak{g}_0)\text{-module}$$

$\mathfrak{g}$ of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)} \Rightarrow \mathfrak{g}_0$ of type $D_n$, $B_n$, $C_n$

sum over $r$ with vertical dominos removed
Example

Type $D^{(1)}_n$, $B^{(1)}_n$, $A^{(2)}_{2n-1}$

$$W^{4,2} \cong W(\begin{array}{c|c}
  &  \\
\hline
  &  \\
\end{array}) \oplus W(\begin{array}{c|c|c}
  & &  \\
\hline
  & &  \\
\end{array}) \oplus W(\begin{array}{c|c|c|c}
  & & &  \\
\hline
  & & &  \\
\end{array})$$

$$\oplus W(\begin{array}{c|c|c}
  & &  \\
\hline
  & &  \\
\end{array}) \oplus W(\begin{array}{c|c|c|c}
  & & &  \\
\hline
  & & &  \\
\end{array}) \oplus W(\emptyset)$$
Dynkin automorphism

Type $A_{n}^{(1)}$:

KKMMNN proved existence of crystals $B_{r,s}$ for $W_{r,s}$
Shimozono gave the combinatorial structure of $B_{r,s}$ using $\sigma$
Dynkin automorphism

Type $A_n^{(1)}$:

KKMMNN proved existence of crystals $B^{r,s}$ for $W^{r,s}$
Shimozono gave the combinatorial structure of $B^{r,s}$
using $\sigma$

e_0 = \sigma^{-1} \circ e_1 \circ \sigma
f_0 = \sigma^{-1} \circ f_1 \circ \sigma
Dynkin automorphism

Type $D^{(1)}_n$:

Okado proved existence of crystals $B^{r,s}$ for $W^{r,s}$

S., Sternberg combinatorial structure of $B^{2,s}$

Sternberg conjecture for $B^{r,s}$

Here we give the combinatorial structure of $B^{r,s}$ for type $D^{(1)}_n$, $B^{(1)}_n$, $A^{(2)}_{2n-1}$ using the Dynkin automorphism $\sigma$
Dynkin automorphism

Type $D^{(1)}_n$: $\sigma$

Type $B^{(1)}_n$: $\sigma$

Type $A^{(1)}_{2n-1}$: $\sigma$

$e_0 = \sigma \circ e_1 \circ \sigma \quad \text{and} \quad f_0 = \sigma \circ f_1 \circ \sigma$
Crystals $B^{1,1}$

**$D^{(1)}_n$**

1 → 2 → ... → n → n-1 → n → n-2 → ... → 2 → 1 → \( \overline{1} \)

**$B^{(1)}_n$**

1 → 2 → ... → n → n → 0 → n → n-1 → 2 → 1 → \( \overline{1} \)

**$A^{(2)}_{2n-1}$**

1 → 2 → ... → n → n → n-1 → 2 → 1 → \( \overline{1} \)
Classical decomposition

By construction

\[ B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \]

as \( X_n = D_n, B_n, C_n \) crystals

\[ \Rightarrow \text{crystal arrows } f_i, e_i \text{ are fixed for } i = 1, 2, \ldots, n \]
Classical crystal

\[ B(\lambda) \subset B([\square])^{\otimes |\lambda|} \]

highest weight

\[
\begin{array}{cccc}
4 & 3 & 2 & 2 \\
2 & 2 & 2 & 1 \\
1 & 1 & 1 & 1
\end{array}
\]

\[ \mapsto 4 \otimes 3 \otimes 2 \otimes 1 \otimes 2 \otimes 1 \otimes 2 \otimes 1 \]

\( f_i, e_i \) for \( i = 1, 2, \ldots, n \) act by tensor product rule

\[
\begin{array}{c}
\varphi_i(b) \\
\varepsilon_i(b)
\end{array}
\quad
\begin{array}{c}
\varphi_i(b') \\
\varepsilon_i(b')
\end{array}
\]
Definition of $\sigma$

$D_n \rightarrow D_{n-1}$ branching

$$B_{D_n}(\Lambda) \cong \bigoplus B_{D_{n-1}}(\text{inner}(P))$$

$\pm$ diagrams $P$

outer$(P) = \Lambda$

$\pm$ diagrams

$\lambda \subset \mu \subset \Lambda$

inner shape

outer shape

$\Lambda/\mu$ horizontal strip filled with $-$

$\mu/\lambda$ horizontal strip filled with $+$
Definition of $\sigma$

$D_{n-1}$ highest weight vectors are in bijection with $\pm$ diagrams via $\Phi$

$$
\Phi:
\begin{array}{|c|c|}
\hline
- & + \\
\hline
+ & - \\
\hline
+ & + \\
\hline
\end{array}
\quad \rightarrow \quad
\begin{array}{llll}
4 & 4 & & \\
2 & 3 & 3 & 1 \\
1 & 1 & 2 & 2 \\
\end{array}
$$
Definition of $\sigma$

$\sigma$ on $\pm$ diagrams

$P \pm$ diagram of shape $\Lambda/\lambda$
columns of height $h$ in $\lambda$

$h \equiv r - 1 \mod 2$:
interchange number of $+$ and $-$ above $\lambda$

$h \equiv r - 1 \mod 2$:
interchange number of $\mp$ and empty above $\lambda$

$P = \begin{array}{ccc}
+ & - & \\
- & + & \\
+ & - & +
\end{array}$

$\mathcal{S}(P) = \begin{array}{cc}
- & \\
- & \\
+ & \\
\end{array}$

$r \geq 6$
$s = 5$
Definition of $\sigma$

$\sigma$ on tableaux

$\quad b \in B^{r,s}$

$e_{\rightarrow a} := e_{a_1} \cdots e_{a_\ell}$ such that $e_{\rightarrow a}(b)$ is $D_{n-1}$ highest weight vector

$\quad f_{\leftarrow a} := f_{a_\ell} \cdots f_{a_1}$

Then

$\sigma(b) = f_{\leftarrow a} \circ \Phi \circ \mathcal{S} \circ \Phi^{-1} \circ e_{\rightarrow a}(b)$
Example

$B^{4,5}$ of type $D_6^{(1)}$

$$b = \begin{array}{ccc}
4 & 4 & \\
3 & 4 & \\
2 & 3 & 1 & 1 \\
1 & 1 & 2 & 3 \\
\end{array} \quad \rightarrow \quad e_4 e_6 e_5 e_4 e_3 e_2 e_2$$

$$\begin{array}{ccc}
4 & 4 & \\
3 & 4 & \\
2 & 3 & 1 & 1 \\
1 & 1 & 2 & 2 \\
\end{array}$$
Example

$B^{4,5}$ of type $D_6^{(1)}$

$b = \begin{array}{ccc}
4 & 4 \\
3 & 4 \\
2 & 3 & 1 & 1 \\
1 & 1 & 2 & 3 \\
\end{array}$

$\Phi^{-1} \rightarrow \begin{array}{ccc}
+ & - \\
+ & \\
- & - \\
+ & \\
\end{array}$

$\rightarrow e_4 e_6 e_5 e_4 e_3 e_2 e_2$

$\rightarrow \begin{array}{cccc}
4 & 4 \\
3 & 4 \\
2 & 3 & 1 & 1 \\
1 & 1 & 2 & 2 \\
\end{array}$

$\rightarrow \begin{array}{ccc}
- & \\
+ & - \\
+ & \\
\end{array}$
Example

\( B^{4,5} \) of type \( D_6^{(1)} \)

\[ b = \begin{array}{ccc}
4 & 4 \\
3 & 4 \\
2 & 3 & 1 & 1 \\
1 & 1 & 2 & 3 \\
\end{array} \]

\[ \Phi^{-1} \rightarrow \begin{array}{ccc}
+ & - \\
+ & & - \\
& - & + \\
\end{array} \]

\[ \Phi \rightarrow \begin{array}{ccc}
3 \\
4 \\
3 & 3 & 3 & 1 \\
1 & 2 & 2 & 2 \\
\end{array} \]

\[ e_4 e_6 e_5 e_4 e_3 e_2 e_2 \rightarrow \begin{array}{ccc}
4 & 4 \\
3 & 4 \\
2 & 3 & 1 & 1 \\
1 & 1 & 2 & 2 \\
\end{array} \]

\[ \subseteq \rightarrow \begin{array}{ccc}
& - \\
& & - \\
& & + \\
\end{array} \]

\[ f_2 f_2 f_3 f_4 f_5 f_6 f_4 \rightarrow \begin{array}{ccc}
2 \\
4 \\
3 & 3 & 4 & 1 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ = \sigma(b) \]
Sketch of Proof

Theorem[FSS]
The KR crystals $B_{r,s}$ of type $D_n^{(1)}$, $B_n^{(1)}$, and $A_{2n-1}^{(2)}$ are uniquely determined by the following properties:

1. As an $X_n$ crystal, $B_{r,s}$ decomposes according as

$$B_{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$$

where $X_n = D_n, B_n, C_n$.

2. $B_{r,s}$ is regular.

3. There is a unique element $u \in B_{r,s}$ such that

$$\varepsilon(u) = s\Lambda_0 \quad \text{and} \quad \varphi(u) = \begin{cases} s\Lambda_0 & \text{for } r \text{ even,} \\ s\Lambda_1 & \text{for } r \text{ odd.} \end{cases}$$

4. $B_{r,s}$ admits the automorphism $\sigma$. 
Sketch of Proof

Theorem [FSS]
The KR crystals $B^{r,s}$ of type $D_n^{(1)}$, $B_n^{(1)}$, and $A_{2n-1}^{(2)}$ are uniquely determined by the following properties:
...

Proof via embedding of Demazure crystal into $B^{r,s}$
⇒ completely fixes 0-arrows
Sketch of Proof

**Condition 1:** Classical decomposition holds by construction.

**Condition 4:** Existence of \( \sigma \) holds by construction.

**Condition 3:** Existence of \( u \) for \( r \) even

\[
u = \emptyset \in B(\emptyset)
\]

\[
\Rightarrow \mathcal{G} \circ \Phi^{-1}(u) = \begin{array}{cccccccc}
- & - & - & - & - & - & - \\
+ & + & + & + & + & + & + \\
\end{array}
\]

\[
\Rightarrow \tilde{u} = \Phi \circ \mathcal{G} \circ \Phi^{-1}(u) = \begin{array}{cccccccc}
2 & 2 & 2 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 \\
\end{array}
\]

\[
\varepsilon(\tilde{u}) = s\Lambda_1 \quad \varphi(\tilde{u}) = s\Lambda_1
\]

\[
\varepsilon(u) = s\Lambda_0 \quad \varphi(u) = s\Lambda_0
\]
Sketch of Proof

**Condition 1:** Classical decomposition holds by construction.

**Condition 4:** Existence of $\sigma$ holds by construction.

**Condition 3:** Existence of $u$ for $r$ odd

\[
\begin{align*}
    u &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \in B(s\omega_1) \\
    \Rightarrow \mathcal{G} \circ \Phi^{-1}(u) &= \begin{bmatrix} \_ & \_ & \_ & \_ & \_ & \_ & \_ \end{bmatrix} \\
    \Rightarrow \tilde{u} &= \Phi \circ \mathcal{G} \circ \Phi^{-1}(u) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \\
    \varepsilon_1(\tilde{u}) &= s \quad \varphi_1(\tilde{u}) = 0 \\
    \varepsilon(u) &= s\Lambda_0 \quad \varphi(u) = s\Lambda_1
\end{align*}
\]
Example

$B^{2,1}$ type $A_5^{(2)}$
Sketch of Proof

**Condition 2:** Regularity of crystal

Need to show: for every $K \subset I = \{0, 1, \ldots, n\}$ with $|K| = 2$ the $K$-component of $B^{r,s}$ is the corresponding $U_q(\mathfrak{g}_K)$-crystal
Sketch of Proof

Condition 2: Regularity of crystal
Need to show: for every $K \subset I = \{0, 1, \ldots, n\}$ with $|K| = 2$ the $K$-component of $B^{r,s}$ is the corresponding $U_q(\mathfrak{g}_K)$-crystal

$K = \{i, j\}$, $i, j \neq 0$ clear by construction
Sketch of Proof

Condition 2: Regularity of crystal

Need to show: for every $K \subset I = \{0, 1, \ldots, n\}$ with $|K| = 2$ the $K$-component of $B^{r,s}$ is the corresponding $U_q(\mathfrak{g}_K)$-crystal

$K = \{0, i\}, i \neq 1$

$$e_0e_i = \sigma e_1\sigma e_i = \sigma(e_1\sigma e_i\sigma)\sigma = \sigma(e_1e_i)\sigma$$
Sketch of Proof

**Condition 2:** Regularity of crystal
Need to show: for every $K \subset I = \{0, 1, \ldots, n\}$ with $|K| = 2$ the $K$-component of $B^{r,s}$ is the corresponding $U_q(\mathfrak{g}_K)$-crystal

$K = \{0, 1\}$ need to show $e_0 e_1 = e_1 e_0$

hard part!!
MuPAD-Combinat...

... is an open source algebraic combinatorics package for the computer algebra system MuPAD

```plaintext
>> KR:=crystals::kirillovReshetikhin(2,2,["D",4,1]):
>> t:=KR([[3],[1]])

```

```
+-----+
| 3   |
+-----+

```

```
>> t::e(0)

```

```
+-----+
| -2  |
+-----+

```

```
+-----+
| 3   |
+-----+
```
MuPAD-Combinat...

... is an open source algebraic combinatorics package for the computer algebra system MuPAD

```plaintext
>> KR:=crystals::kirillovReshetikhin(2,2,["D",4,1]):
>> t:=KR([[3],[1]])

+----+----+
| -2 | -1 |
+----+----+

>> t::sigma()

+--------+
| -2 | -1 |
+--------+

| 2 | 3 |
+--------+
```
Open Problems

- Existence and combinatorial structure for other KR crystals $C_n^{(1)}$, $D_{n+1}^{(2)}$, ...
- Characterization of unrestricted rigged configurations (done for type $A_n^{(1)}$)
- Fermionic formulas for unrestricted Kostka polynomials
  Relation to fermionic formulas of [HKKOTY]?
- Relation to other rigged configurations [S]
  $\leadsto$ LLT polynomials
- Relation to box ball systems, description in terms of R-matrices
- Level restriction