An approach through combinatorics to intersection number questions for the moduli spaces of curves

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Outline

I Maps in orientable and non-orientable surfaces
  • Basic facts about embeddings, and examples
  • An algebraic encoding of maps

II The map series and two examples
  • Jack symmetric functions and an integral representation
  • Two models from mathematical physics
  • Aspects of the moduli spaces of algebraic curves

III Maps and the moduli space of smooth curves
  • Faber’s Top Intersection Number Conjecture
  • An approach for all genera through localisation and algebraic combinatorics
Part I
Maps in orientable and non-orientable surfaces
Two maps are said to be equivalent if one can be obtained from the other by smooth deformations of the surface.

The cyclic order of edges around a vertex is invariant under smooth deformations of the surface.
1.2 - The Embedding Theorem

Encode vertex degrees by $[2 \ 2 \ 2 \ 3 \ 3]$.

Encode face degrees by $[3 \ 4 \ 5]$. 

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An anticlockwise circulation around each vertex can be consistently defined

For example, the anticlockwise circulation around this vertex is

\[(4^+ 6^- 5^-)\]

- Let \( \nu = (5^+ 3^+) \cdot (1^- 3^-) \cdot (1^+ 2^- 4^-) \cdot (2^+ 6^+) \cdot (4^+ 6^- 5^-) \in \mathfrak{S}_12, \) the vertex permutation for the map
- Let \( \epsilon = (1^+ 1^-) \cdots (6^+ 6^-) \in \mathfrak{S}_12, \) the edge permutation of the map

**Theorem (Embedding).** The vertex permutation \( \nu \) uniquely defines a map. The faces of the map correspond to the cycles of the face permutation \( \nu \epsilon \)
Demonstration of the use of the Embedding Theorem

- Let $\nu = (5^+ 3^+) \cdot (1^- 3^-) \cdot (1^+ 2^- 4^-) \cdot (2^+ 6^+) \cdot (4^+ 6^- 5^-) \in \mathcal{S}_{12}$, the vertex permutation for the map: cycle-type $[2^3 3^2] \vdash 12$. This is also the vertex-type of the map.

- Then $\phi := \nu \varepsilon = (1^+ 3^- 5^+ 4^+) \cdot (2^+ 4^- 6^-) \cdot (2^- 6^+ 5^- 3^+ 1^-) \in \mathcal{S}_{12}$ is the face permutation of the map: cycle-type $[3^4 5] \vdash 12$
Part II
The map series and two applications
II.1 - The maps series

Definition. The map series is the formal sum

\[ M^O_A(u, x, y, z) = \sum_{m \in M^O_A} u^{g(m)} x^{\#v(m)} y^{\#f(m)} z^{\#e(m)} \in \mathbb{C}[u, x, y] [[z]] \]

where

- \( M^O_A \) is the set of all maps in orientable surfaces with face degrees in the set \( A \)
- \( \#v(m) \) is the number of vertices of the map \( m \) etc.; \( g(m) \) is the genus of \( m \)

- The coefficient \([u^g x^i y^j z^k]M^O_A\) of \( u^g x^i y^j z^k \) in \( M^O_A \) is the number of maps in \( M^O_A \) with genus \( g \), \( i \) vertices, \( j \) faces, \( k \) edges
Theorem. Let $\Pi_A$ be the set of all partitions with parts in $A$, and let $C_\alpha$ be the conjugacy class of $G_{2n}$ indexed by $\alpha + 2n$.

$$M^O_A(u^2, x, y, z) = 2u^2 z \frac{\partial}{\partial z} \log R_A^O(xu^{-1}, yu^{-1}, uz/2)$$

where

$$R_A^O(x, y, z) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\nu, \phi \neq 2n, \phi \in \Pi_A} |C_{\nu \epsilon} \cap C_{\phi}| [x]^{l(\nu)} [y]^{l(\phi)}$$

- Embedding Theorem - vertex partition $\nu$
- Remove decoration - labelling and orienting of edges
- Retain only connected maps - this device was known to Hurwitz
- Euler's formula - $\#_\nu(m) - \#_e(m) + \#_f(m) = 2 - 2g(m)$
• $\mathbb{C}\mathfrak{S}_{2n}$ denotes the group algebra of $\mathfrak{S}_{2n}$ over $\mathbb{C}$

• For $\alpha \vdash 2n$, let

\[ K_\alpha = \sum_{\pi \in \mathcal{C}_\alpha} \pi \]

• The centre $Z_{\mathbb{C}\mathfrak{S}_{2n}}$ of $\mathbb{C}\mathfrak{S}_{2n}$ is spanned by $K_\alpha$ for $\alpha \vdash 2n$.

• Let $\nu, \phi \vdash 2n$. Then

\[ |C_\nu \cap C_\phi| = \frac{|C_\phi|}{|C_{[2n]}|} [K_\phi] K_\nu K_{[2n]} \]

• $Z_{\mathbb{C}\mathfrak{S}_{2n}}$ has a basis consisting of orthogonal idempotents

• The orthogonal idempotents are linear combinations of the $K_\alpha$ with scalars that are evaluations of characters $\chi^\theta$ of irreducible representations of $\mathfrak{S}_{2n}$.

• We use the orthogonal idempotents to determine $[K_\phi] K_\nu K_{[2n]}$ and thence the combinatorial number $|C_\nu \cap C_\phi|$.
Theorem. The map series for orientable surfaces is

\[ M_A^O(u^2, x, y, z) = 2u^2 z \frac{\partial}{\partial z} \log R_A^O(xu^{-1}, yu^{-1}, uz/2) \]

where

\[ R_A^O(x, y, z) = \sum_{n \geq 0} \frac{z^n}{n!(2n)!} \sum_{\phi \in \Pi_A} |C_\phi|^{|l(\phi)} \sum_{\theta \vdash 2n} x_\phi \chi_\theta \theta |_{2^n} H_\theta(x) \]

and

- \( \chi_\phi^\theta \) is the evaluation of the character \( \chi^\theta \) on the conjugacy class \( C_\phi \)
- \( H_\theta(x) = \prod_{1 \leq i \leq l(\theta)} (x - i + 1)^{\theta_i} \) where \( \theta = (\theta_1, \theta_2, \ldots) \), and
- \( (x)^{(k)} = x(x + 1) \cdots (x + k - 1) \)
- \( \Pi_A \) is the set of all partitions with parts in \( A \)
II.2 - Quadrangulations and partition functions

Corollary. Let $M_4^O$ be the map series for quadrangulations in orientable surfaces. Let $M$ be the map series of all maps in orientable surfaces. Then

$$M_4^O(u^2, x, y, z) = \frac{1}{2}(M^O(4u^2, x + u, x, yz^2) + M^O(4u^2, x - u, x, yz^2))$$

- The proof is character theoretic
- No constructive proof is known
- A similar result holds for triangulations

- Two brief comments connected with string theory
  - $M_4^O$ and $M^O$ are partition functions associated with quantum chromodynamics and 2-d quantum gravity, before the scaling limit is taken
  - The former involves quark-antiquark pairs as time evolves, taking account of the interaction of gluons
II.3 - Extension to hypermaps in all surfaces

The algebraic theory is more symmetrical for hypermaps.

A hypermap with:
- Vertex-type: $[2^2 2^6 6] \vdash 34$
- Hyperedge-type: $[3^3 4^2] \vdash 17$
- Face-type: $[3^3 4^2] \vdash 17$

- Vertex-type $[2^2 4^6 6] \vdash 34$ is encoded algebraically as $x_1^2 x_4^6 x_6^1$
- Hyperedge-type $[3^3 4^2] \vdash 17$ is encoded as $z_3^3 z_4^2$
- Face-type $[3^3 4^2] \vdash 17$ is encoded as $y_3^3 y_4^2$
- The corresponding group is the Double Coset Algebra $S_{2n} \backslash S_n / S_{2n}$ of the hyperoctahedral group $S_{2n}$. 

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- \( p_k = x_1^k + x_2^k + \cdots \) and \( p_{[a,b,...]} = p_a p_b \cdots \) (power sums)

- Jack symmetric functions \( J_{\alpha}(x,a) \) are orthogonal with respect to
  \[
  \langle p_\alpha, p_\beta \rangle_a = \frac{|\alpha|!}{c_\alpha} a^{l(\alpha)} \delta_{\alpha,\beta}
  \]

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**Theorem.** Let \( \Psi \equiv \Psi(p(x), p(y), p(z); b) \) where

\[
\Psi = (1 + b) t \frac{\partial}{\partial t} \log \sum_\theta \frac{t^{l(\theta)}}{|| J_\theta ||_{1+b}} J_\theta(x; 1+b) J_\theta(y; 1+b) J_\theta(z; 1+b) \bigg|_{t=1}
\]

Then

\[
H^O(x, y, z) = \Psi(x, y, z; 0), \quad \text{hypermap series for orientable surfaces},
\]

\[
H(x, y, z) = \Psi(x, y, z; 1), \quad \text{hypermap series for all surfaces}.
\]

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**Conjecture.** Then \( \Psi(x, y, z, b) \) is the hypermap series where \( b \) marks an invariant of hypermaps
Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ and $a$ is an indeterminate. Let

$$\langle \cdot \rangle^a_{\mathbb{R}} : f \rightarrow \frac{\int_{\mathbb{R}^N} f(\lambda) |V(\lambda)|^{2a} e^{-\frac{a}{2} p_2(\lambda)} d\lambda}{\int_{\mathbb{R}^N} |V(\lambda)|^{2a} e^{-\frac{a}{2} p_2(\lambda)} d\lambda}$$

Then

$$[p_2^m] J_\theta(\lambda, a^{-1}) = \frac{\langle J_\theta(\lambda, a^{-1}) \rangle^a_N}{J_\theta(1_N, a^{-1})}$$

**Theorem.** Recall that $b$ marks a conjectural invariant of maps. Then

$$M(1, x, N, z; b) = \frac{2}{1 + b} \partial \log \left\langle \frac{1}{z} xy \sum_{k \geq 1} \frac{1}{k} z^{k/2} x p_k(\lambda) \right\rangle^{1+b}_N$$

$$\in \mathbb{Q}[x, N, b][[z]]$$

$M(1, x, y, z; b) \in \mathbb{Q}[x, y, b][[z]]$ is got by replacing $N$ by $y$ to mark faces.
II.4 - Example from the moduli space of curves

- Use Strebel quadratic derivative to obtain a cell decomposition of the moduli space of curves
- This gives its virtual Euler characteristic as a weighted sum of monopoles (maps with 1 vertex)
- The integration is carried out by the Sel'berg integration theorems

Conjecture. The conjectural map invariant marked by $b$ has a geometric interpretation on the moduli space of curves
Part III
Faber’s Top Intersection Number Conjecture
Using this theorem we have:

\textbf{Theorem (GSV)}

\[ K_{g}^{\theta} = (-1)^{k} \left< \tau_{a_{1}} \tau_{a_{2}} \cdots \tau_{a_{k}} \right> \]  

\text{(Witten symbol)}

where

\[ \left< \tau_{a_{1}} \tau_{a_{2}} \cdots \tau_{a_{k}} \right> := \int_{M_{g,n}} \psi^{a_{1}} \cdots \psi^{a_{m}} \lambda_{k} \]

\( (\theta = (a_{1}, \ldots, a_{m}) \).

\[ \square \]

\textbf{Corollary (GSV) \quad Polynomiality}

\[ \left< \tau_{a_{1}} \tau_{a_{2}} \cdots \lambda_{k} \right> = (-1)^{k} \left[ a_{1}^{a_{1}} \cdots a_{m}^{a_{m}} \right] P_{m}^{g}(a_{1}, \ldots, a_{m}) \]

where

\[ P_{m}^{g} \] is a polynomial in \( a_{1}, \ldots, a_{m} \) with terms of degree between \( 2g-2+m \) and \( 3g-3+m = \text{dim} \ M_{g,n} \).

\[ \square \]

We shall use this result in the Double Hurwitz Problem, although indirectly.
The ELSV theorem and its significance

**Theorem (Ekedahl, Lands, Shaprio and Vainshtein)**

$$H^g_{2g} = C(g, u) \int_{H_{g,m}} \frac{1 - \lambda_1 x_1 - \cdots - (-1)^g \lambda_g}{\prod_{i=1}^m (1 - x_i \psi_i)}$$

where

$$C(g, u) = u! \prod_{i=1}^m \frac{x_i}{x_i!}$$

and

- $\overline{M}_{g,m}$: Deligne-Mumford compactification of the moduli space of genus $g$ curves with $m$ marked points
- $\psi_i$: a certain codimension 1 class ($= c_1(L_i)$)
  where
  - $c_j$ is the $j$-th Chern class
  - $L_i$ is a natural line bundle (cotangent space to the $i$-th marked point)
- $\lambda_g$: a certain codimension $g$ class (Chern(cohomology) class)
  ($= c_g(E)$) where
  - $E$ is a natural rank $g$ vector bundle.

**Note:** Only the general form of this result will be needed in this talk.
III.1 - The conjecture

- Let $\overline{M}_{g,n}^{\text{rt}}$ be the compactification of the moduli space of genus $g$ smooth curves (curves with "rational tails") with $n$ marked points
- The intersection number for $\overline{M}_{g,n}^{\text{rt}}$ is denoted by

$$\langle \tau_{a_1} \cdots \tau_{a_n} \lambda_k \rangle_g^{\text{rt}} = \langle \psi_1^{a_1} \cdots \psi_n^{a_n} \lambda_k \rangle_g^{\text{rt}}$$

where $a_1 + \cdots + a_n - k = g - 2 + n$ and $\psi_i$ is a 1-dimensional Chow class and $\lambda_k$ is a $k$-dimensional Chow class
- Top intersection numbers correspond to $k = 0$

Conjecture (Faber). Let $g \geq 3$, $a_1, \ldots, a_n \geq 1$, $a_1 + \cdots + a_n = g + n - 2$. Then

$$\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g^{\text{rt}} = \frac{(g - 3 + n)!(2g - 1)!!}{n!} \frac{(g - 1)! \prod_{i=1}^{n} (2a_i - 1)!!}{(g - 1)! \prod_{i=1}^{n} (2a_i - 1)!!}$$

where $k!! = 1 \cdot 3 \cdot 5 \cdots k$ for $k$ odd
Approach I

- Getzler and Pandharipande proved that Faber's Top Intersection Number Conjecture is a consequence of the Virasoro Conjecture for $\mathbb{P}^2$
- Givental has outlined a proof of the Virasoro Conjecture for $\mathbb{P}^m$
- Thus a proof of Faber's Top Intersection Number Conjecture seems to be almost complete for (all $g$ and all $n$)
- The proof is lengthy, dense and indirect

Approach II

- We (Goulden, DMRJ and Vakil) propose an alternative approach that we believe to be more direct
- The proof holds for all $g$ (genera)
- The approach is through localisation theory from the work of Atiyah and Bott
III.2 - Outline of the proposed proof

- We use localisation theory to express the Conjecture in terms of
localisation trees, a tree weighted by Hurwitz Numbers, and the top
potential

- A Hurwitz Number counts ramified covers of the sphere by a curve of
genus $g$ with prescribed ramification over $\infty$ (and 0 in the case of the
Double Hurwitz Number) and all other branch points elementary

- A ramified cover can be encoded as a transitive ordered factorisation
of a permutation into transpositions

- The top potential

$$P_n^g(\alpha) = \sum_{a_1, \ldots, a_n \geq 0, n - \delta_{g, 1} \leq a_1 \leq \ldots \leq a_n \leq g + n - 2} \left\langle \tau_{a_1} \cdots \tau_{a_n} \right\rangle_g \alpha_1^{a_1} \cdots \alpha_n^{a_n}$$

satisfies a topological recursion

- We re-express Faber's Top Intersection Number Conjecture in terms of a
formal differential system with underlying Lagrangian structures
### III.3 - A localisation tree

**Edge weights:** integers > 0
- $\varepsilon_i$: on $0\infty$-edges
- $c_i$: on $\infty1$-edges

**Partitions**
- $\beta^1 = (\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_5)$ etc
- $\gamma_i = (c_i^1, \ldots, c_i^{k_i})$ etc
- $\delta^0 = (\varepsilon_1, \varepsilon_6)$ etc
- Balance: $|\beta^i| = |\gamma^i|

**Vertex weights:**
- $H_{\beta, \gamma}^0$: Hurwitz Number; ramification $\beta$ at 0 & $\gamma$ at $\infty$
- $H_{\delta}^0$: Hurwitz Number; ramification $\delta^i$ at $\infty$
- $P_n^0(\delta^0)$: Top potential at $g$-vertex

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0-vertices $\quad \infty$-vertices $\quad 1$-vertices
A ramified covers of the sphere

- $X$: $d$-sheeted source curve
- $P_1, \ldots, P_r$: simple branch points
- $\tau_1, \pi, \sigma$ sheet transitions
- $\langle \tau_1, \ldots, \tau_r, \pi \rangle$ acts transitively on sheet labels $\{1, \ldots, d\}$ so $X$ is connected
- $\tau_1 \cdots \tau_r \pi \sigma = \iota$: path from base point is contractible
- $g$: genus of $X$

- In the example: Ignore objects in dotted box
  - $d = 5$ sheets; $r = 5$ simple branch points
  - $\tau_1 = (1, 2), \tau_2 = (2, 5), \tau_3 = (1, 4), \tau_4 = (3, 5), \tau_5 = (4, 5)$
  - $\pi = (1, 3, 4)(2, 5)$
  - $\infty$: ramification type $[3, 2]$
Theorem. The genus 0 Double Hurwitz Series $H^0$ defined by

$$H^0(z, u : p, q) = \sum_{\alpha, \beta \in \mathcal{P}} \frac{H^0_{\alpha, \beta}}{r^0_{\alpha, \beta}! |\text{Aut } \alpha| |\text{Aut } \beta|} \frac{p_\alpha q_\beta z^{|\beta|} u^{|(\beta)|}}{|(\beta)|}$$

satisfies the Join-Cut Equation

$$\sum_{i \geq 1} p_i \frac{\partial H^0}{\partial p_i} + u \frac{\partial H^0}{\partial u} - 2H^0 = \sum_{i, j \geq 1} \left( \frac{i \cdot j}{2} p_{i+j} \frac{\partial H^0}{\partial p_i} \frac{\partial H^0}{\partial p_j} + \frac{i + j}{2} p_i p_j \frac{\partial H^0}{\partial p_{i+j}} \right)$$

Definition. The weighted potential $\Psi^g$ is

$$\Psi^g(z, u; p) = \sum_{n \geq 1} \sum_{t \in \mathcal{T}_n} \frac{P^g_{\alpha,(\delta^0)}}{M! |\text{Aut } \alpha|} \prod_{m \in \delta^0} \frac{m^m}{m!} \prod_{i} \varepsilon_i \prod_{j} \frac{H^0_{\delta^0_j, (1)^{|\delta^0_j|}}}{r^{\delta^0_j}_0! |\delta^0_j|!} \prod_{k} \frac{H^0_{\beta^k, \gamma_k}}{r_k^{\gamma_k}! \delta^0_k!} u^{\gamma_k} - 2z^{|\beta^0_k|}$$

where the sum is over all genus $g$ localisation trees

- The terms on the RHS correspond to:  
  - $g$-vertex.  
  - $\infty$-vertices,  
  - non-root 0-vertices,  
  - $g$-vertex,  
  - $0$-$\infty$-edges
III.4 - The top intersection numbers $\langle \tau_{a_1} \cdots \tau_{a_n} \rangle^g_{rt}$

Theorem. Let $\xi^{(i)}(z, u; p) = \sum_{j \geq 1} \frac{j^{j+i}}{j!} f_j(z, u; p)$ where $f_j$ is given by

$$
\begin{align*}
    f_j &= u^{-2} \left( j \frac{\partial}{\partial q_j} H^0(z, u; p; q) \right) \bigg|_{q_i = g_i, i \geq 1}, \quad j \geq 1, \\
    g_j &= \left( j \frac{\partial}{\partial q_j} H^0(1, 1; e_1; q) \right) \bigg|_{q_i = f_i, i \geq 1}, \quad j \geq 1
\end{align*}
$$

Then $\Psi^g(z, u; p) = \sum_{n \geq 1} \frac{1}{n!} \sum_{a_1, \ldots, a_n \geq 0} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle^g_{rt} \prod_{j=1}^n \xi^{(a_j)}(z, u; p)$

- Let $F^g$ be the solution of the partial differential equation

$$
\Delta F^g = \sum_{i,j \geq 1} \left( p_{i+j} \left( p_i^* \widehat{H}^0 \right) p_j^* + \frac{p_i p_j}{2} p_{i+j}^* \right) F^g + \sum_{i \geq 1} z^{2g} p_i p_i^* \widehat{H}^0
$$

where $\Delta = z \frac{\partial}{\partial z} - 1 + \sum_{i \geq 1} p_i \frac{\partial}{\partial p_i}$ and $p_i^* = k \partial / \partial p_k$ (adjoint action)

Then

$$
c_g F^g(z; p) = [u^{2g-1}] \Psi^g(z(1-u)^{-1}, -u; u(1-u)^{-1} p)
$$
III.5 - Reduction to a polynomial identity

- For a specific number of parts, this theorem reduces the proof of Faber's Top Intersection Number Conjecture to checking that two polynomials are identical.

- The proof
  - is direct
  - holds for all genera $g$
  - holds for $n = 1, 2, 3$ at the moment

- Comments
  - We know the Double Hurwitz Numbers $H_{\alpha,\beta}^0$ for all $\alpha$ with $l(\alpha) \leq 5$ and for all $\beta$
  - We believe that we understand how to use the Join-Cut Equation for $H^0$ to avoid determining $H_{\alpha,\beta}$ explicitly.
  - This part of the work is in progress