This week’s lecture will cover Chapter 3 and Chapter 4 Sections 3.1 – 4.2

Chapter 3 Linear Equations of Higher Order (Page # 144)

Section 3.1 Introduction: Second Order Linear Equations (Page # 144)

A second order differential equation in the (unknown) function $y(x)$ is one of the form $G(x, y, y', y'') = 0$. This differential equation is said to be linear provided that $G$ is linear in the dependent variable $y$ and its derivatives $y', y''$.

The DE $e^x y'' + (\cos x) y' + (1 + \sqrt{x}) y = \tan^{-1} x$ is a second order linear DE. On the other hand $e^x y'' + (\cos x) y y' + (1 + \sqrt{x}) y = \tan^{-1} x$ is not a second order linear FE because of the non linear term $yy'$. The general second order linear DE has the form

$$A(x) y'' + B(x) y' + C(x) y = F(x) \quad \text{.........(a)}$$

Homogeneous Second Order Linear Equations: Consider the general second order linear equation $A(x) y'' + B(x) y' + C(x) y = F(x)$, where the coefficient functions $A, B, C$ and $F$ are continuous on the open interval $I$, and if the function $F$ on the right hand side vanishes identically on $I$ then the equation (a) is said to be homogeneous linear equation; otherwise it is non-homogeneous.

The second order linear equation $x^2 y'' + 2xy' + 3y = \cos x$ is non-homogeneous; its associated homogeneous equation is $x^2 y'' + 2xy' + 3y = 0$. Note that the non-homogeneous term $F(x,y)$ frequently corresponds to some external influence on the system.

Example 1. Verify that the functions $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions of the differential equation $y'' - 2y' + y = 0$ and then find a solution satisfying the initial conditions $y(0) = 3$ and $y'(0) = 1$.

Solution: You can easily verify that $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions of the differential equation $y'' - 2y' + y = 0$. Now by the superposition principle (Page# 146, Theorem 1) we know that the general solution is $y(x) = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 xe^x$. We also have now $y'(x) = (c_1 + c_2)e^x + c_2 xe^x$.

The following results are the found:

$$y(0) = c_1 = 3 \quad y'(0) = c_1 + c_2 = 1$$

$$\Rightarrow c_1 = 3, \quad c_2 = -2$$
**Definition**: Linear Independence of two functions: Two functions defined on an open interval I is said to linearly independent on I provided that neither is a constant multiple of the other.

**Definition**: Given two functions \( p \) and \( q \), the Wronskian \( W(p, q) \) of \( p \) and \( q \) is the determinant

\[
W(x) = W(p, q) = \begin{vmatrix}
p & q \\
p' & q'
\end{vmatrix} = pq' - p'q
\]

**Theorem**: Wronskians of solutions: Suppose that \( y_1 \) and \( y_2 \) are two solutions of the homogeneous second order linear equation \( y'' + p(x)y' + q(x)y = 0 \) on a open interval \( I \), on which \( p \) and \( q \) are continuous.

a) If \( y_1 \) and \( y_2 \) are linearly dependent, then the Wronskian \( W(y_1, y_2) \equiv 0 \) on \( I \).

b) If \( y_1 \) and \( y_2 \) are linearly independent, then the Wronskian \( W(y_1, y_2) \neq 0 \) at each point of \( I \).

**Home work problems**: (Page # 155) 1, 3, 6, 8, 10, 16, 17, 32, 51

**Section 3.2 General Solutions of Linear Equations** (Page # 158)

**Definition**: Linear Dependence of Functions: The \( n \) functions \( f_1, f_2, f_3, \ldots, f_n \) are said to be linearly dependent on \( I \) provided that there exists constants \( c_1, c_2, c_3, \ldots, c_n \) not all zero such that \( c_1f_1 + c_2f_2 + c_3f_3 + \cdots + c_nf_n = 0 \) on \( I \) for all \( x \).

**Example 1**: Show that the functions \( f_1(x) = \sin 2x, \ f_2(x) = \sin x \cos x, \ f_3(x) = e^x \) are linearly dependent by using Wronskian.

**Example 2**: Show that the functions \( f_1(x) = e^{-3x}, \ f_2(x) = \cos 2x, \ f_3(x) = \sin 2x \) are linearly independent.

**Example 3**: Show that \( y(x) = -3e^{-3x} + 3\cos 2x - 2\sin 2x \) is a solution of \( y''' + 3y'' + 4y' + 12y = 0 \).

**Home work problems**: (Page # 167) 1, 3, 5, 8, 10, 13, 17, 20, 39

**Section 3.3 Homogeneous Equations with Constant Coefficients** (Page # 170)

**Characteristic equation for finding general solution**: Suppose \( y = e^{rx} \) be a solution of the homogeneous equation \( a_ny^n + a_{n-1}y^{n-1} + \cdots + a_2y' + a_1y + a_0y = 0 \), with constant coefficients \( a_0, a_1, a_2, \ldots, a_n \), then \( a_nr^n + a_{n-1}r^{n-1} + \cdots + a_2r^2 + a_1r + a_0 = 0 \) is called the characteristic equation or auxiliary equation of the DE. The solution of DE is reduces to a solution of a purely algebraic equation.
Example 1. Solve the initial value problem
\[ y'''' + 3y''' - 10y' = 0; \quad y(0) = 7, \quad y'(0) = 0, \quad y''(0) = 70 \]

Solution: Let \( y = e^{rx} \) be the solution of the initial value problem. The characteristic equation of this DE is \( r^4 + 3r^3 - 10r = 0 \Rightarrow r = -5, \ 0, \ 2 \). By the above theorem the general solution is \( y = c_1 + c_2e^{-5x} + c_3e^{2x} \). Using initial condition one can find that \( c_1 = 0, \ c_2 = 2, \ c_3 = 5 \). The particular solution is then \( y(x) = 2e^{-5x} + 5e^{2x} \).

Home work problems: (Page # 180) 1, 3, 5, 8, 10, 13, 17, 25, 39

Section 3.5 Nonhomogeneous Equations and Undetermined Coefficient (Page # 195)

The general nonhomogeneous \( n \)-th order linear equation with constant coefficients has the form
\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(x) \ldots \ (b) \]
has the general solution of the form \( y = y_c + y_p \), where the complementary function \( y_c(x) \) is a general solution of the associated homogeneous equation
\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0 \]
and \( y_p(x) \) is the particular solution of (b). For finding the particular solution of (b) we need to make an intelligent guess. Let find particular solution:

Example 1. Find the particular solution \( y_p(x) \) of \( y'''' + 3y''' + 4y = 3x + 2 \). Our DE is nonhomogeneous of the form (b) where \( f(x) = 3x + 2 \) is polynomial of degree 1, so our guess is \( y_p(x) = Ax + B \), then \( y'_p = A, \ y''_p = 0 \) will satisfy the DE provided that
\[ 0 + 3A + 4(Ax + B) = 3x + 2 \Rightarrow A = 3/4, \ B = -1/16 \]. We have the particular solution \( y_p(x) = 3/4x - 1/16 \).

Section 3.6 Forced Oscillations and Resonance (Page # 209)

In this section we have the second order DE \( mx'' + cx' + kx = F(t) \) that governs the one dimensional motion of mass \( m \) that is attached to a spring (with constant \( k \)) and a dashpot (with constant \( c \)) and is also acted on by an external force \( F(t) \). Machines with rotating components commonly involve mass-spring system (or their equivalents) in which the external force is simple harmonic: \( F(t) = F_0 \cos \omega t \) or \( F(t) = F_0 \sin \omega t \), where the constant \( F_0 \) is the amplitude of the periodic force and \( \omega \) is its circular frequency.

Undamped force oscillations: To study the undamped oscillations under the influence of the external force \( F(t) = F_0 \cos \omega t \) and setting \( c = 0 \), we have \( mx'' + kx = F_0 \cos \omega t \). Note that it has complementary function \( x_c(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t \) when \( \omega_0 = \sqrt{\frac{k}{m}} \) is the
(circular) natural frequency of the mass-spring system. One can find the particular solution of the system equal to \( x_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \).

**Example 1.** Consider the DE \( mx'' + kx = F(t) \). Also given that \( m = 1, k = 9, F_0 = 80, \) and \( \omega = 5 \). Find the general solution \( x(t) \) if \( x(0) = x'(0) = 0 \).

### Section 3.8 Endpoint Problems and Eigenvalues (Page # 229)

We are familiar with the fact that a solution of a second-order linear DE is uniquely determined by two initial conditions. In this section we will see that the situation is radically different for a problem such as \( y''' + p(x)y' + q(x)y = 0; \ y(a) = 0, \ y(b) = 0 \). The given conditions are at the end points of the interval \((a, b)\). Such a problem is called an endpoint or boundary value problem.

In particular \( y''' + p(x)y' + q(x)y = 0; \ y(a) = 0, \ y(b) = 0 \) has the trivial solution \( y(0) = 0 \).

**Example 1.** Consider the boundary value problem \( y''' + 3y = 0; \ y(0) = 0, \ y(\pi) = 0 \). It is easy to see that the general solution is \( y(x) = A\cos x\sqrt{3} + B\sin x\sqrt{3} \). Using end points we find \( A = 0, \ B = 0 \). Thus the only trivial solution is \( y(x) = 0 \).

**Example 2.** Consider the boundary value problem \( y''' + 4y = 0; \ y(0) = 0, \ y(\pi) = 0 \). It is easy to see that the general solution is \( y(x) = A\cos 2x + B\sin 2x \). Using end points we find \( A = 0, \) and for all values of \( B \) we have the nontrivial solution is \( y(x) = B\sin(2x) \).

### Chapter 4 Introduction to Systems of Differential Equations (Page # 242)

### Section 4.1 First Order Systems and Applications (Page # 242)

In this section we will restrict our attention to systems in which the number of equations is the same as the number of dependent variables.

First order systems: In case of a system of two second order equations we consider the form

\[
x_1'' = f_1(t, x_1, x_2, x_1', x_2'), \quad x_2'' = f_2(t, x_1, x_2, x_1', x_2')
\]

The first order system consisting of the single \( n \)-th order equation

\[
x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}).
\]

where we introduce the following dependent variables

\[
x_1 = x, \quad x_2 = x', \quad x_3 = x'', \quad \ldots, \quad x_n = x^{(n-1)}.
\]

**Example 1.** Consider the third order system of linear DE \( x^{(3)} + 3x'' + 2x' - 5x = \sin 2t \).

This problem is of the form \( f(t, x, x', x'') = 5x - 2x' - 3x'' + \sin 2t \). The substitutions will
be \( x_1 = x, \ x_2 = x' = x'_1, \ x_3 = x'' = x'_2 \Rightarrow x'_1 = x_2, \ x'_2 = x_1, \ x'_3 = 5x_1 - 2x_2 - 3x_3 + \sin 2t \) of the three first order equations.

**Example 2.** Consider the second order system of linear DE
\[
2x'' = -6x + 2y \\
y'' = 2x - 2y + 40 \sin 3t
\]
Transfer this system into an equivalent first order system.
We use \( x_1 = x, \ x_2 = x' = x'_1, \ y_1 = y, \ y_2 = y' = y'_1 \) to obtain the following
\[
\begin{align*}
x'_1 &= x_2 \\
x'_2 &= -6x_1 + 2y_1 \\
y'_1 &= y_2 \\
y'_2 &= 2x_1 - 2y_1 + 40 \sin 3t
\end{align*}
\]
of first order equations in the dependent variables \( x_1, \ x_2, \ y_1, \ y_2 \).

**Section 4.2 The Method of Elimination** (Page # 254)

The method of elimination for linear differential systems of equations is similar to the solution of a linear system of algebraic equations by a process of eliminating the unknowns at a time only a single equation with a single unknown remains.

**Example 1** Find the particular solution of the system \( x' = 4x - 3y, \ y' = 6x - 7y \) that satisfies the initial conditions \( x(0) = 2, \ y(0) = -1 \).

By the method of elimination we find \( x = \frac{1}{6} y' + \frac{7}{6} y \Rightarrow x' = \frac{1}{6} y'' + \frac{7}{6} y' \). Thus we have \( x' = 4x - 3y \Rightarrow \frac{1}{6} y'' + \frac{7}{6} y' = 4\left( \frac{1}{6} y' + \frac{7}{6} y \right) - 3y \Rightarrow y'' + 3y' - 10y = 0 \). Using characteristic equation one can find the general solution of the homogeneous system as \( y(t) = C_1 e^{2t} + C_2 e^{-5t} \Rightarrow x(t) = \frac{1}{6} (2C_1 e^{2t} - 5C_2 e^{-5t}) + \frac{6}{7} (C_1 e^{2t} + C_2 e^{-5t}) \). Now using initial condition we have the desired solution \( x(t) = 3e^{2t} - e^{-5t}, \ y(t) = 2e^{2t} - 3e^{-5t} \).