Section 0.1 Polynomial and Rational Functions

Polynomials The combination of algebraic expressions is a polynomial, such as \( x^2 - 5x + 6, 3x^2 - 27, y^3 - 5y - 6, x \). A polynomial is a single term or sum of terms with variables having whole number exponents. General form of a polynomial is

\[
P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0.
\]

Where the real numbers \( a_n, a_{n-1}, \cdots, a_1, a_0 \) are the coefficients, and \( n \) is a non-negative integer. For a polynomial of degree \( n \), we must have \( a_n \neq 0 \), is the leading coefficient and \( a_0 \) is the constant term. The domain of a polynomial is the entire real line.

Rational Functions Like the rational numbers \( \frac{3}{7}, \frac{1}{2}, \frac{3}{1} \), the expressions \( \frac{3x-1}{7-x}, \frac{1-x^2}{2x-4}, \frac{3}{1-2x} \) are the rational expressions. The combination of rational expressions where numerator and denominator are both polynomials are called rational functions. The general form of rational functions is

\[
R(x) = \frac{p(x)}{q(x)}, q(x) \neq 0
\]

where \( p(x), q(x) \neq 0 \) are polynomial functions. The domain of a rational function is the set of all real values of \( x \) for which the denominator is not zero.

Equations and Inequalities Expressions connected with the equality sign \( = \) forms an equation, while expressions connected with one of the following inequality signs \(<, >, \leq, \geq \) forms inequalities.

Linear equations and linear inequalities
Equation of the form \( ax + b = 0, a \neq 0 \) is a linear equation. The only solution of this equation is \( x = -\frac{b}{a}, a \neq 0 \). On the other hand an example of a linear inequality could be \( ax + b \geq 0, a \neq 0 \) has the solution lies in the interval \( [-b/a, \infty) \). Remember that the solution of a linear equation is a single point on the real line whereas the solution of a linear inequality consists of points in an interval.

Quadratic equations and quadratic inequalities
Equation of the form \( ax^2 + bx + c = 0, a \neq 0 \) is a quadratic equation, which has no more than two real solutions namely \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \). If possible, with rational zeros, using factors we can solve a quadratic equation. In this section we will provide examples of
factoring quadratic expressions and thereby solve the quadratic equations and inequalities.

**Factoring quadratic expressions**

Examples

Case 1: Leading coefficient 1

1. Factor $x^2 - 5x + 6 = (x - 2)(x - 3)$, as -2, -3 are factors of 6 and adds up to -5
2. Factor $x^2 - 5x - 6 = (x - 6)(x + 1)$, as -6, 1 are factors of -6 and adds up to -5

Case 2: Leading coefficient is not 1

1. Factor $2x^2 - 5x + 3 = (\Box x - 3)\left(x - \frac{2}{\Box}\right) = (2x - 3)(x - 1)$, as -2, -3 are factors of 6 ($= 2 \times 3$) and adds up to -5. We insert leading coefficient 2 in the boxes.
2. Factor $6x^2 - 7x + 2 = (\Box x - 3)\left(x - \frac{4}{\Box}\right) = (6x - 3)(x - 2/3) = (2x - 1)(3x - 2)$, as -3, and -4 are factors of 12 ($= 6 \times 2$) and adds up to -7. We insert leading coefficient 6 in the boxes and perform necessary algebraic works.

Case 3: Factors by grouping

1. Factor $x^4 - x^3 - x + 1$

\[
x^4 - x^3 - x + 1 = x^3(x - 1) - 1(x - 1) = (x - 1)(x^3 - 1)
= (x - 1)(x - 1)(x^2 + x + 1)
\]

Remember the formula $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

Case 4: Using formula $x^2 - a^2 = (x - a)(x + a)$

1. Factor $x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$

**Absolute value functions**

For any real number $x$, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

**Properties of $|x|$**

1. $|ab| = |a||b|$ and, if $b \neq 0$, $\frac{|a|}{|b|} = \frac{a}{b}$
2. $|a + b| \leq |a| + |b|$
3. $|x| < a$ if and only if $-a < x < a$
4. $|x| > a \geq 0$ if and only if $x < -a$ or $x > a$
Note: Properties 3 and 4 are true for $\leq, \geq$ as well. The function $|x|$ is always even and has y-axis symmetry.

Examples on equations and inequalities

Solve the following equations:
1. $8x + 3 = 67$  \hspace{1cm} \text{Answer: } x = 8
2. $\frac{x - 4}{x + 1} = 0$  \hspace{1cm} \text{Answer: } x = 4
3. Solve $\frac{1}{2}(x + 5) - 4 = \frac{1}{3}(2x - 1)$  \hspace{1cm} \text{Answer: } x = -7
4. Solve $\frac{3}{x - 2} = \frac{1}{x - 1} + \frac{7}{(x - 1)(x - 2)}$  \hspace{1cm} \text{Answer: } x = 4

Solve the following inequalities and write answers in interval(s), use real line test:
1. Solve the inequality $9 < 1 + 4x \leq 13$ and write your answer in interval notation.
2. Solve $\frac{x - 1}{x + 2} \leq 0$
3. Solve $x^2 + x - 6 \geq 0$
4. Solve $3 - 2x \geq -7$
5. Solve $x^2 - 5x - 6 \geq 0$
6. Solve $x^2 - 5x + 6 \geq 0$
7. Solve $x^2 + x - 6 \leq 0$
8. Solve $x^2 - x - 6 \leq 0$
9. Solve $x^2 - 5x + 6 \leq 0$
10. Solve $\frac{x^2 - 5x + 6}{x - 1} \geq 0$
11. Solve $\frac{x^2 - 5x + 6}{x - 1} \leq 0$
12. Solve $\frac{x^2 - 5x + 6}{x - 1} > 0$
13. Solve $\frac{x^2 - 5x + 6}{x - 2} \geq 0$
14. Solve $\frac{x - 3}{x - 1} \geq 2$

Hint: Do not cross multiply to solve. Use $\frac{x - 3}{x - 1} - 2 \geq 0 \Rightarrow -\frac{x + 1}{x - 1} \geq 0$ etc.

15. Solve $|x - 1| \leq 3$
16. Solve $|x^2 - 5x| \leq 6$
17. Solve \( |x^2 - 5x| \geq 6 \)

18. Plot the rational function \( \frac{x^2 - 9}{x^2 - 1} \) showing all asymptotes.

19. Find the domain of \( f(x) = \sqrt[3]{x^2 - 5x + 6} \)

20. Find all zeros of the polynomial \( P(x) = x^3 - 1 \)

21. Find the point(s) of intersection of the curves \( y = x^2 - x - 5, \ y = x + 3 \)

22. Find the slope of the line through the points (3, 5) and (7, -9)

23. Find the equation of a straight line through (5, 7) and (-3, 1)

Section 0.2 Graphing Calculators and CAS (Computer Algebra System)

1. Plot the functions \( f(x) = |x|, \ g(x) = \frac{|x|}{x} \) on the same window. You may use either your calculator or Maple.

2. Plot the rational function \( P(x) = \frac{x^2 - 5x + 6}{x - 2} \) showing all asymptotes and roots if any.

3. Plot the rational function \( P(x) = \frac{x - 2}{x - 2} \)

4. Plot the rational function \( P(x) = \frac{x - 2}{x - 3} \)

5. Plot the rational function \( P(x) = \frac{x - 2}{x^2 + 2} \)

6. Use graphing devise to estimate zeros of \( P(x) = x^4 - 7x^3 - 15x^2 - 10x - 1410 \)

7. Use graphing devise to estimate zeros of \( P(x) = x^3 - 20x^2 - x + 20 \)

Section 0.3 Inverse Functions

**Definition** A function \( f(x) \) is said to be one-to-one when for every \( y \in Range\{f\} \), there is exactly one \( x \in Domain\{f\} \) for which \( y = f(x) \). By horizontal line test we can check one-to-one function if there is no more than one point of contact with the graph of \( f(x) \).

Two functions \( f(x) \) and \( g(x) \) are said to be inverse to each other if \( (f \circ g)(x) = (g \circ f)(x) = x \). Note that \( (f \circ g)(x) = f(g(x)) \) is the composition of \( f \) with \( g \)

**Theorem** A function \( f(x) \) has an inverse if and only if it is one-to-one.

Examples
1. Test whether $f(x) = x^3$ is one-to-one. If it is, find inverse.
2. Show that $f(x) = x^4$ is not one-to-one in its domain. Find its inverse restricting the domain.
3. For the given function $f(x) = x^3 - 5$, determine $f^{-1}(x)$ and $[f(x)]^{-1}$.
4. Use the calculator, domain and range concept to draw the graph of $f(x) = x^5 + 8x^3 + x + 1$ and its inverse.
5. Determine algebraically the inverse of $f(x) = \sqrt{x^3 + 1}$. Draw the graphs.
6. Graph the inverse function of $f(x) = \frac{x}{\sqrt{x^2 + 4}}$.
7. For $f(x) = \frac{1}{x + 2}$ and $g(x) = \frac{1 - 2x}{x}$ show that $(f \circ g)(x) = (g \circ f)(x) = x$. What can you say about the functions?
8. Given $f(x) = x^3 + 4x - 1$, find $f^{-1}(-1)$ without finding inverse.

**Section 0.4 Trigonometric and Inverse Trigonometric Functions**

$$\cos^2(x) + \sin^2(x) = 1 \quad 1 + \tan^2(x) = \sec^2(x) \quad \cot^2(x) + 1 = \csc^2(x)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$
$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$
$$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$$
$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \begin{cases} 
\cos^2(x) - \sin^2(x) \\
2\cos^2(x) - 1 \\
1 - 2\sin^2(x) 
\end{cases}$$
$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$
$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$
$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

$$\cos \left( \frac{x}{2} \right) = \pm \sqrt{\frac{1 + \cos(x)}{2}}$$
$$\sin \left( \frac{x}{2} \right) = \pm \sqrt{\frac{1 - \cos(x)}{2}}$$
$$\tan \left( \frac{x}{2} \right) = \pm \sqrt{\frac{1 - \cos(x)}{1 + \cos(x)}}$$
\[
\sin(x)\sin(y) = \frac{1}{2}[\cos(x - y) - \cos(x + y)]
\]
\[
\cos(x)\cos(y) = \frac{1}{2}[\cos(x - y) + \cos(x + y)]
\]
\[
\sin(x)\cos(y) = \frac{1}{2}[\sin(x + y) + \sin(x - y)]
\]
\[
\cos(x)\sin(y) = \frac{1}{2}[\sin(x + y) - \sin(x - y)]
\]

\[
\sin(x) + \sin(y) = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)
\]
\[
\sin(x) - \sin(y) = 2\sin\left(\frac{x - y}{2}\right)\cos\left(\frac{x + y}{2}\right)
\]
\[
\cos(x) + \cos(y) = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)
\]
\[
\cos(x) - \cos(y) = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)
\]

For two vectors \( \mathbf{A} \) and \( \mathbf{B} \), \( \mathbf{A} \cdot \mathbf{B} = ||\mathbf{A}|| ||\mathbf{B}|| \cos(\theta) \)

The well known results: \( \text{soh}, \text{cah}, \text{toa} \)

\( \text{soh} \): \( s \) stands for sine, \( o \) stands for opposite and \( h \) stands for hypotenuse, \( \sin x = \frac{o}{h} \)

\( \text{cah} \): \( c \) stands for cosine, \( a \) stands for adjacent \( h \) stands for hypotenuse, \( \cos x = \frac{a}{h} \)

\( \text{toa} \): \( t \) stands for tan, \( o \) stands for opposite and \( a \) stands for adjacent, \( \tan x = \frac{o}{a} \).

Where \( x \) is the angle between the hypotenuse and the adjacent.

Other three trigonometric functions have the following relations:

\[
\csc x = \frac{1}{\sin x} = \frac{h}{o}, \quad \sec x = \frac{1}{\cos x} = \frac{h}{a} \quad \text{and} \quad \cot x = \frac{1}{\tan x} = \frac{a}{o}
\]

\( \sin(n\pi \pm x) = [\ ?] \sin x, \ \cos(n\pi \pm x) = [\ ?] \cos x, \ \tan(n\pi \pm x) = [\ ?] \tan x \), the sign \( ? \) is for plus or minus depending on the position of the terminal side. One may remember the four-quadrant rule: (All Students Take Calculus: A = all, S = sine, T = tan, C = cosine)

\[
\begin{array}{c|c}
\text{sine} & \text{all} \\
\hline
\text{tan} & \text{cosine}
\end{array}
\]
Example: Find the value of \( \sin 300^\circ \). We may write

\[
\sin 300^\circ = \sin(2 \cdot 180^\circ - 60^\circ) = [-\sin 60^\circ] = -\frac{\sqrt{3}}{2},
\]

in this case the terminal side is in quadrant four where sine is negative.

In the following diagram, each point on the unit circle is labeled first with its coordinates (exact values), then with the angle in degrees, then with the angle in radians. Points in the lower hemisphere have both positive and negative angles marked.
Important values:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>$30^\circ = \frac{\pi}{6}$</th>
<th>$45^\circ = \frac{\pi}{4}$</th>
<th>$60^\circ = \frac{\pi}{3}$</th>
<th>$90^\circ = \frac{\pi}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>cos</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>tan</td>
<td>0</td>
<td>$\frac{1}{\sqrt{3}}$</td>
<td>1</td>
<td>$\sqrt{3}$</td>
<td>Undefined</td>
</tr>
<tr>
<td>csc</td>
<td>Undefined</td>
<td>2</td>
<td>$\frac{2}{\sqrt{3}}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>sec</td>
<td>1</td>
<td>$\frac{2}{\sqrt{3}}$</td>
<td>$\sqrt{2}$</td>
<td>2</td>
<td>Undefined</td>
</tr>
<tr>
<td>cot</td>
<td>Undefined</td>
<td>$\sqrt{3}$</td>
<td>1</td>
<td>$\frac{1}{\sqrt{3}}$</td>
<td>0</td>
</tr>
</tbody>
</table>

The functions \(\sin\) and \(\cos\) are periodic of period \(2\pi\), so that
\[
\sin(2\pi + x) = \sin x, \quad \cos(2\pi + x) = \cos x
\]

On the other hand \(\tan\) is periodic of period \(\pi\), thus \(\tan(\pi + x) = x\)

As a general form it is always true that \(\sin(2n\pi + x) = \sin x, \quad \cos(2n\pi + x) = \cos x\) and \(\tan(n\pi + x) = x\), for \(n\) is a positive integer. Also
\[
\sin(\pi / 2 + x) = \cos x, \quad \cos(\pi / 2 + x) = \sin x
\]

**Solution of trigonometric functions**

1. Find all solutions of the equation \(2\sin x - \sqrt{3} = 0\), for any integer \(n\).
   Solution:
   
   \[
   2\sin x - \sqrt{3} = 0
   \]
   
   \[
   \sin x = \frac{\sqrt{3}}{2} = \sin(\pi / 3) = \sin(2\pi / 3)
   \]
   
   Then \(x = 2n\pi + \pi / 3\) or \(x = 2n\pi + 2\pi / 3\)

2. Find all solutions of the equation \(2\cos^2 x + 8\cos x - 10 = 0\), for any integer \(n\).
   Solution: Factor the equation to
   
   \[
   (\cos x + 5)(\cos x - 1) = 0
   \]
   
   \[
   \cos x = -1 = \cos(0), \quad \cos x + 5 \neq 0
   \]
   
   Then \(x = 2n\pi\)
3. Find all solutions of the equation $2\sin^2 x + \sqrt{3}\sin x = 0$, for any integer $n$.
Solution: Factor the equation to 
$$\sin x(\sin x - \sqrt{3}/2) = 0,$$
$$\sin x = 0 = \sin(0) = \sin(\pi),$$
and
$$\sin x = \sqrt{3}/2 = \sin(\pi/3) = \sin(2\pi/3).$$
Then $x = n\pi$ or $x = 2n\pi + \pi/3$ or $x = 2n\pi + 2\pi/3$
Here we have two diagrams, one is given and another one is in Example 1.

4. Find all solutions of the equation $2\sin x - \sqrt{3} = 0$, for any integer $n$.

**Trigonometric sum of angles**

For any real numbers $x$ and $y$, the following identities hold:

1. $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$
2. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

When $x = y$ we have the following important identities

1. $\sin 2x = 2\sin x \cos x$
2. $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$ and also $\sin^2 x = 1/2(1 - \cos 2x)$, $\cos^2 x = 1/2(1 + \cos 2x)$

**Sum to product form**

3. $\sin x + \sin y = 2\sin \left( \frac{x+y}{2} \right) \cos \left( \frac{x-y}{2} \right)$
4. If you replace $y$ by $-y$ you get $\sin x - \sin y = 2\sin \left( \frac{x-y}{2} \right) \cos \left( \frac{x+y}{2} \right)$
5. $\cos x + \cos y = 2\cos \left( \frac{x+y}{2} \right) \cos \left( \frac{x-y}{2} \right)$
6. $\cos x - \cos y = 2\sin \left( \frac{x+y}{2} \right) \sin \left( \frac{y-x}{2} \right)$

**Inverse Trigonometric functions (Remember the relations)**

1. If $y = \sin^{-1} x$ then $x = \sin y$ and $y \in [-\pi/2, \pi/2]$
   If $\sin(\sin^{-1} x) = x$ then $x \in [-1, 1]$
If \( \sin^{-1}(\sin x) = x \) then \( x \in [-\pi/2, \pi/2] \)

2. If \( y = \cos^{-1} x \) then \( x = \cos y \) and \( y \in [0, \pi] \)
   - If \( \cos(\cos^{-1} x) = x \) then \( x \in [-1, 1] \)
   - If \( \cos^{-1}(\cos x) = x \) then \( x \in [0, \pi] \)

3. If \( y = \tan^{-1} x \) then \( x = \tan y \) and \( y \in (-\pi/2, \pi/2) \)
   - If \( \tan(\tan^{-1} x) = x \) then \( x \in (-\infty, \infty) \)
   - If \( \tan^{-1}(\tan x) = x \) then \( x \in (-\pi/2, \pi/2) \)

4. If \( y = \sec^{-1} x \) then \( x = \sec y \) and \( y \in [0,\pi/2) \cup (\pi/2, \pi] \)

**Section 0.5 Exponential and Logarithmic Functions**

Remember the following formulas

1. **Hyperbolic functions**
   a) \( \sinh x = \frac{e^x - e^{-x}}{2} \)
   b) \( \cosh x = \frac{e^x + e^{-x}}{2} \)

2. **Logarithmic functions: \( x > 0, y > 0 \)**
   a) \( \log x := \log_{10} x, \ln x := \log_e x \)
   b) \( \log(xy) = \log x + \log y, \ln(xy) = \ln x + \ln y \)
   c) \( \log(x/y) = \log x - \log y, \ln(x/y) = \ln x - \ln y \)
   d) \( \log x = y \iff 10^y = x, \ln x = y \iff e^y = x \)
   e) \( \log x^p = p \log x \)
   f) \( \log_m m = 1, m > 0; \log 1 = 0, \ln 1 = 0 \)
   g) \( \log_y x = \frac{\log x}{\log y} = \frac{\ln x}{\ln y} \)

3. Solve the following equations for \( x \):
   a) \( e^{4x} = e^8 \)
   b) \( xe^{-2x} + 2e^{-2x} = 0 \)
   c) \( x^2 \ln x - 9 \ln x = 0 \)
   d) \( \ln(x+1) = 8 \)
   e) \( e^{4x+3} = 5 \)
   f) \( 8e^{2\ln x} = 512 \)

4. Write the following result as a single logarithmic expression
a) \( \ln 25 - 3 \ln(1/5) \)

b) \( \ln(5/8) + 3 \ln(2) - \ln 5 \)

5. Determine the following values

a) \( \log_5 25 \)  
b) \( \log_5 (1/25) \)  
c) \( \log_2 (32) \)

6. Round your answer to two decimal places if necessary

a) Given \( f(x) = 3 \sinh(3x) \), find \( f(2) \)

b) Given \( f(x) = -9 \cosh(x) \), find \( f(2) \)

c) Given \( f(x) = ae^{ax} \), find \( x \) when \( f(0) = 2 \), \( f(2) = 14 \), \( f(x) = 20 \)

### Section 0.6 Transformations of functions

Formulas to understand and to keep in mind (Consider \( c > 0 \))

Compare the following graphs with the graph of \( y = f(x) \)

1. The graph of \( y = -f(x) \) is the reflection along x-axis

2. The graph of \( y = f(-x) \) is the reflection along y-axis

3. The graph of \( y = f(x) + c \) has vertical translation (shift) of \( c \) units upward

4. The graph of \( y = f(x) - c \) has vertical translation (shift) of \( c \) units downward

5. The graph of \( y = f(x - c) \) has horizontal translation (shift) of \( c \) units to the right

6. The graph of \( y = f(x + c) \) has horizontal translation (shift) of \( c \) units to the left

Vertical Stretch VSt (or elongate) Versus Shrink VSh (or compress).

Suppose the number \( c \) is positive. When \( c \) is negative the graph has horizontal reflection.

7. The graph of \( y = cf(x) \) has vertical stretch by a factor of \( c \) units if \( c > 1 \) and has vertical shrink by a factor of \( c \) when \( c < 1 \).

8. The graph of \( y = f(cx) \) has horizontal shrink by a factor of \( c \) units if \( c > 1 \) and has horizontal stretch by a factor of \( c \) when \( c < 1 \)
Examples

1. Find the domain of the following functions

a) \( f(x) = \sqrt{x - 3} \)  
b) \( f(x) = \sqrt{x^2 - 3x + 4} \)  
c) \( f(x) = \sqrt{x - 10} + \sqrt{x} \)  
d) \( f(x) = \sqrt{x^2 - 9} \)  
e) \( f(x) = \frac{\sqrt{x^2 - 9}}{x - 3} \)

3. Discuss the transformation of the following functions comparing with the graph of \( y = f(x) \)

a) \( y = 8x^2 - 5, \ f(x) = x^2 \)  
b) \( y = x^2 + 12x + 27, \ f(x) = x^2 \)  
c) \( f(x) = 2(x - 3)^3 + 7, \ f(x) = x^3 \)  
d) \( f(x) = 2(3x - 3)^3 + 7, \ f(x) = x^3 \)