2-3: System of equations and matrices

Systems, Matrices, and Applications
Systems of Linear Equations

System of equation

(Has solution) Consistent  Inconsistent (has no solution)

Dependent  Independent

For Example: Consider the system

\[ \begin{align*}
3x + 2y &= -1 \\
5x - 3y &= 11
\end{align*} \]

Solve it and see that it has a unique solution. The system is consistent and independent.

The system

\[ \begin{align*}
x - y &= 4 \\
2x - 2y &= 3
\end{align*} \]

Is inconsistent, it has no solution. Check the solution.

Now the system

\[ \begin{align*}
x - y &= 4 \\
2x - 2y &= 8
\end{align*} \]

has infinitely many solutions, it is consistent and dependent.

We can solve it by the following methods:

1. **Algebraically Using** Elimination
2. **Algebraically Using** Substitution
3. **Using** Graphical method
4. **Augmented Matrices and Row Operations.**
5. **Matrix algebra (the \( A^{-1}B \) form)**
6. **Cramer’s Rule (Determinants).**

**Definition:** The following representation of two linear equations is called the system of linear equations in two variables:

\[ \begin{align*}
x_1 + 2x_2 &= 3 \\
2x_1 - x_2 &= 1
\end{align*} \]
**Solution:** A solution of the given system is an ordered set or list of numbers that satisfies all the equations simultaneously when we put \( x_1 = a \) and \( x_2 = b \). This solution is written as \( (x_1, x_2) = (a, b) \). We have our solution in this case as \( (x_1, x_2) = (1, 1) \).

If the given system has at least one solution, it is said to be **consistent**. When the system has no solution, it is said to be **inconsistent**.

**Examples**

1. The system \( x_1 + 2x_2 = 3 \), \( 2x_1 - x_2 = 1 \) has unique solution. Find the solution.
2. The system \( x_1 + x_2 = 2 + 2x_3 \), \( -x_1 + 3x_2 = -3 \) has infinitely many solutions. Write the solution in terms of the third variable.
3. The system \( 2x_1 + 2x_2 = 2 \), \( x_1 + x_2 = -3 \) is inconsistent.
4. Solve the system \( 4x_1 + 3x_2 = 10 \), \( -3x_1 + 2x_2 = 18 \)
5. If the system \( 2x_1 + kx_3 = 5 \), \( x_1 + 3x_2 = 7 \) is inconsistent find \( k \).
6. Solve the system \( x_1 + 2x_2 + x_3 = 4 \), \( -x_1 + 3x_2 + 2x_3 = 4 \), \( 2x_1 - 4x_2 + x_3 = -1 \)
7. Consider this 2 by 2 system: use all the methods discussed above to solve the system:

\[
\begin{align*}
3x + 2y & = -1 \\
5x - 3y & = 11
\end{align*}
\]

**Solution:**

a) **Algebraically Using** Elimination

\[
\begin{align*}
3(3x + 2y = -1) & \implies 9x + 6y = -3 \\
2(5x - 3y = 11) & \implies 10x - 6y = 22
\end{align*}
\]

By substituting \( x = 1 \) we get \( y = -2 \)

b) **Algebraically Using** Substitution

\[
3x + 2y = -1 \implies y = -3/2 x - 1/2
\]

*So we can substitute \( y = -3/2 x - 1/2 \) in other equation*

\[
5x - 3 -3/2 x - 1/2 = 11 \implies x = 1
\]

c) **Using** Graphical method

\[
\begin{align*}
3x + 2y & = -1 \implies y = -\frac{3}{2} x - \frac{1}{3} \\
5x - 3y & = 11 \implies y = \frac{5}{3} x - \frac{11}{3}
\end{align*}
\]
### d) Augmented Matrices and Row Operations.

\[
\begin{align*}
3x + 2y &= -1 \\
5x - 3y &= 11
\end{align*}
\]

\[
\begin{bmatrix}
3 & 2 & -1 \\
5 & -3 & 11
\end{bmatrix} \Rightarrow
\begin{bmatrix}
1 & 2/3 & -1/3 \\
5 & -3 & 11
\end{bmatrix} \Rightarrow
\begin{bmatrix}
1 & 2/3 & -1/3 \\
0 & -19/3 & 38/3
\end{bmatrix} \Rightarrow
\begin{bmatrix}
1 & 2/3 & -1/3 \\
0 & 1 & -2
\end{bmatrix}
\]

The last row gives \( y = -2 \), and by backward substitute we get \( x = 1 \)

Answers: 1. \( (1, 1) \) 2. \( (x_1 = 3 - 5x_3 / 3, x_2 = -1 + x_3 / 3, x_3 = x_3) \) 3. No solution 4. \( (-2, 6) \)
5. \( k = 6 \) 6. \( (1, 1, 1) \)

#### Example 2.
Let’s see another example by using Augmented Matrices and Row Operations. Solve for \( x, y \) and \( z \)

\[
\begin{align*}
x - 2y &= 6 - 4z \\
x + 13z &= 6 - y \\
-2x + 6y - z &= -1
\end{align*}
\]

or

\[
\begin{align*}
x - 2y - 4z &= 6 \\
x + y + 13z &= 6 \\
-2x + 6y - z &= -1
\end{align*}
\]

**Augmented Matrix:**

\[
\begin{bmatrix}
1 & -2 & 4 & 6 \\
1 & 1 & 13 & 6 \\
-2 & 6 & -1 & -1
\end{bmatrix}
\]

Multiply row 1 by \(-1\) and add with row 2 and get your new row 2.

\[
\begin{bmatrix}
1 & -2 & 4 & 6 \\
0 & 3 & 9 & 0 \\
-2 & 6 & -1 & -1
\end{bmatrix}
\]

\( R_2' = R_2 - 1R_1 \)

Multiply 2 with row 1 and add with row 3 and get your new row 3.

\[
\begin{bmatrix}
1 & -2 & 4 & 6 \\
0 & 3 & 9 & 0 \\
0 & 2 & 7 & 11
\end{bmatrix}
\]

\( R_3' = R_3 + 2R_1 \)

Multiply row 2 by \(1/3\) gives you following.

\[
\begin{bmatrix}
1 & -2 & 4 & 6 \\
0 & 1 & 3 & 0 \\
0 & 2 & 7 & 11
\end{bmatrix}
\]

\( R_2' = \frac{1}{3} R_2 \)

Multiply by \(-2\) with row 2 and add with row 3 and get your row 3.

\[
\begin{bmatrix}
1 & -2 & 4 & 6 \\
0 & 1 & 3 & 0 \\
0 & 0 & 1 & 11
\end{bmatrix}
\]

\( R_3' = R_3 - 2R_2 \)
Now by backward substitution to solve for variables:

\[
\begin{align*}
z &= 11, \quad y + 3(11) = 0 \quad x - 2(-6) + 4(11) = 6 \\
y &= -33 \quad x &= -104
\end{align*}
\]

\[
\begin{bmatrix}
2 & 1 & -1 & 2 \\
1 & 3 & 2 & 1 \\
1 & 1 & 1 & 2
\end{bmatrix}
\]

Now practice: Write the system of equations and solve. Ans (2, -1, 1)

**Reduced Row Echelon Form** (rref):

The Elimination Method for solving large systems of linear equations

1. Make the leading coefficient 1 either by interchanging row or by multiplying or dividing the first by a suitable constant.
2. Eliminate the leading coefficient each later equation by replacing the later equation by the sum itself and a suitable multiple of the first equation
3. Repeat step 1 and 2 for the 2nd row to eliminate leading coefficient 0 and to make 2nd element 1
4. Continue the process until all diagonal elements are 1 then do back substitution to solve variables.

**Example 3.** Below are three row-reduced echelon forms for matrices of certain linear systems. For each matrix, tell how many solutions the system has. Explain. If there are any, find the solutions. If there are infinitely many solutions, find the general formula and two particular solutions.

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Solutions</th>
</tr>
</thead>
</table>
| a. \[
\begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 5
\end{bmatrix}
\] | The system has unique solution with x=3, y=-2, z=5 |
| b. \[
\begin{bmatrix}
1 & 0 & -2 & 0 \\
0 & 1 & 4 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] | The system does not have solution |
| c. \[
\begin{bmatrix}
1 & 0 & -2 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] | The system has many solutions, it is dependent and consistent. The general solutions are \[y + 3z = 0\] and \[x - 2z = 0\] and \[y = -3z\] and \[x = 2z\] |
Now let’s discuss system of equations again:

1. **Matrix algebra (the $A^{-1}B$ form)**

Example: let

\[ 3x = 9 \quad \Rightarrow \quad x = \frac{9}{3} = 3^{-1}(9) = 3 \]

For matrix algebra we also use this inverse concept to solve systems of equation. **Singular Matrix**: Singular matrices do not have an inverse. A matrix is singular matrix if determinant of the matrix is equal to zero, let $A$ is a matrix then $A^{-1}$ exist if $|A| \neq 0$.

**So let’s discuss how to find determinant of a matrix**

**Finding determinant**

If

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad |A| = ad - bc \quad \text{and} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]

Example: $A = \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}$ then $|A| = (-9 - 10) = -19 \neq 0$

Therefore $A$ is not singular matrix. It has inverse. Find the inverse. And verify that $A^{-1}A = AA^{-1} = I$. What is $I$? Answer: $I$ is an identity matrix.

**Finding Determinant of a 3 by 3 matrix**

\[ A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & 1 & 13 \\ -2 & 6 & -1 \end{bmatrix} \]

\[ |A| = \begin{vmatrix} 1 & 13 \\ 6 & -1 \end{vmatrix} -(-2) \begin{vmatrix} 1 & 13 \\ -2 & -1 \end{vmatrix} + 4 \begin{vmatrix} 1 & 1 \\ -2 & 6 \end{vmatrix} \\
= 1(-1 - 78) + 2(-1 - (-26)) + 4(6 - (-2)) = 3 \]

Note: The determinant of a matrix will be zero if

1. An entire row is zero.
2. Two rows or columns are equal.
3. A row or column is a constant multiple of another row or column.

Remember, that a matrix is invertible, non-singular, if and only if the determinant is not zero. So, if the determinant is zero, the matrix is singular and does not have an inverse.
**Sarrus Rule.** In order to compute

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}
\]

Note: Sarrus rule is only applicable if the determinant is of order 3 by 3.

Example 1. Use Sarrus rule to find the value of \( A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \)

\[
2 \times 3 + 4 + 48 = 59
\]

\[
4 \times 2 + 4 + 32 = 59
\]

\[
A = 59 - 64 = -5
\]

Exercise 1. Use Sarrus rule to evaluate the determinant \( A = \begin{bmatrix} 8 & 3 & 4 \\ 4 & 6 & 1 \\ 5 & 2 & 4 \end{bmatrix} \) and verify your answer using calculator.

**Finding inverse of a matrix:**

Now let’s learn how to find inverse of a matrix. There are different methods to find inverse matrix.

**Method 1.** Use Shortcut for 2 by 2 matrix

Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) then \( A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \)

Example:

\[
A = \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} \text{ then } A^{-1} = \frac{1}{3(-3)-5(2)} \begin{bmatrix} -3 & -2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}
\]
Method 2 (Optional). Use Gauss-Jordan elimination to transform \([ A | I ]\) into \([ I | A^{-1} ]\).

**Example:** Consider a matrix \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) and write the following (use \texttt{rref} using calculator)

\[
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ -2/3 & 1/3 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}
\]

Method 3 (Optional). **Adjoint** method

\[ A^{-1} = \frac{1}{\det A} (\text{adjoint of } A) \quad \text{or} \quad A^{-1} = \frac{1}{\det A} (\text{cofactor matrix of } A)^T \]

Let \( A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \), then \(|A| = -5\) and \( A^{-1} = \frac{1}{-5} \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix} \)

Now we know how to find inverse, let’s go back to solution of system of equations:

**Example 1.** (continued) Given system is \( \begin{cases} 3x + 2y = -1 \\ 5x - 3y = 11 \end{cases} \)

If we write in matrix form then we get the following,

\[
A = \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix} = B = -1 \\
11
\]

If we do \( AX = B \), we get the given system and we can rewrite

\[
AX = B \\
\Rightarrow X = A^{-1}B
\]
Now, we have seen that \( A = \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} \) then \( A^{-1} = \frac{1}{3(-3) - 5(2)} \begin{bmatrix} -3 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{19} & \frac{2}{19} \\ \frac{5}{19} & -\frac{3}{19} \end{bmatrix} \)

Now
\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{19} & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}
\]
so \( x = 1 \) and \( y = -2 \)

**Cramer’s Rule (Determinants)**

Given the system

\[
a_1x + b_1y = c_1
\]
\[
a_2x + b_2y = c_2
\]

This system has the unique solution

\[
x = \frac{D_x}{D} \quad \text{and} \quad y = \frac{D_y}{D}
\]

\[
D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \quad \text{and} \quad D \neq 0
\]

**Example 1. (Continued) Solve (using Cramer’s Rule)**

\[
3x + 2y = -1
\]
\[
5x - 3y = 11
\]

\[
D = \begin{vmatrix} 3 & 2 \\ 5 & -3 \end{vmatrix} = (-9 - 10) = -19 \quad D_x = \begin{vmatrix} -1 & 2 \\ 11 & -3 \end{vmatrix} = (3 - 22) = -19
\]
\[
D_y = \begin{vmatrix} 3 & -1 \\ 5 & 11 \end{vmatrix} = (33 + 5) = 38
\]

Now, \( x = \frac{D_x}{D} = \frac{-19}{-19} = 1 \) and \( y = \frac{D_y}{D} = \frac{38}{-19} = -2 \)
Example 2. Use Cramer’s Rule to solve the system:

\[4x - y + z = -5\]
\[2x + 2y + 3z = 10\]
\[5x - 2y + 6z = 1\]

Solution. We begin by setting up four determinants:

\[D, D_x, D_y, \text{and } D_z:\]

\(D\) consists of the coefficients of \(x, y,\) and \(z\) from the three equations.

\[
D = \begin{vmatrix}
4 & -1 & 1 \\
2 & 2 & 3 \\
5 & -2 & 6
\end{vmatrix}
\]

\(D_x\) is obtained by replacing the \(x\)-coefficients in the first column of \(D\) with the constants from the right sides of the equations.

\[
D_x = \begin{vmatrix}
-5 & -1 & 1 \\
10 & 2 & 3 \\
1 & -2 & 6
\end{vmatrix}
\]

\(D_y\) is obtained by replacing the \(y\)-coefficients in the second column of \(D\) with the constants from the right sides of the equations.

\[
D_y = \begin{vmatrix}
4 & -5 & 1 \\
2 & 10 & 3 \\
5 & 1 & 6
\end{vmatrix}
\]

\(D_z\) is obtained by replacing the \(z\)-coefficients in the third column of \(D\) with the constants from the right sides of the equations.

\[
D_z = \begin{vmatrix}
4 & -1 & -5 \\
2 & 2 & 10 \\
5 & -2 & 1
\end{vmatrix}
\]

Next, we evaluate the four determinants:

\[
D = \begin{vmatrix}
4 & -1 & 1 \\
2 & 2 & 3 \\
5 & -2 & 6
\end{vmatrix}
= 4(12 - (-6)) + 1(12 - 15) + 1(-4 - 10)
= 4(18) + 1(-3) + 1(-14)
= 72 - 3 - 14
= 55
\]
\[ D_x = \begin{vmatrix} -5 & -1 & 1 \\ 10 & 2 & 3 \\ 1 & -2 & 6 \end{vmatrix} = -5 \begin{vmatrix} 2 & 3 \\ -2 & 6 \end{vmatrix} - (-1) \begin{vmatrix} 10 & 3 \\ 1 & 6 \end{vmatrix} + 1 \begin{vmatrix} 10 & 2 \\ 1 & -2 \end{vmatrix} \]

\[ = -5(18) - 1(57) + 1(-22) \]

\[ = -90 + 57 - 22 \]

\[ = -55 \]

\[ D_y = \begin{vmatrix} 4 & -5 & 1 \\ 2 & 10 & 3 \\ 5 & 1 & 6 \end{vmatrix} = 4 \begin{vmatrix} 10 & 3 \\ 1 & 6 \end{vmatrix} - (-5) \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} + 1 \begin{vmatrix} 10 & 2 \\ 5 & 1 \end{vmatrix} \]

\[ = 4(60 - 3) + 5(12 - 15) + 1(2 - 50) \]

\[ = 4(57) + 5(-3) + 1(-48) \]

\[ = 228 - 15 - 48 \]

\[ = 165 \]

\[ D_z = \begin{vmatrix} 4 & -1 & -5 \\ 2 & 2 & 10 \\ 5 & -2 & 1 \end{vmatrix} = 4 \begin{vmatrix} 2 & 10 \\ -2 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 2 & 10 \\ 5 & 1 \end{vmatrix} + (-5) \begin{vmatrix} 2 & 2 \\ 5 & -2 \end{vmatrix} \]

\[ = 4(22) + 1(-48) - 5(-4 - 10) \]

\[ = 4(22) + 1(-48) - 5(-14) \]

\[ = 88 - 48 + 70 \]

\[ = 110 \]

Substitute these four values into the formula from Cramer’s Rule:

\[
\begin{align*}
    x &= \frac{D_x}{D} = \frac{-55}{55} = -1 \\
    y &= \frac{D_y}{D} = \frac{165}{55} = 3 \\
    z &= \frac{D_z}{D} = \frac{110}{55} = 2
\end{align*}
\]

So, the solution is \((-1, 3, 2)\).
Matrix Algebra on the Calculator TI:
1. Hit \textbf{2nd-MATRIX}, then \textbf{EDIT}, and select \([A], [B], \text{etc.}\)
2. Type in the size(order) of the matrix and the entries. Remember to hit \textbf{ENTER} after each entry.
3. Repeat these steps until you have all your matrices entered.
4. To calculate hit \textbf{2nd-QUIT}, then hit \textbf{2nd-MATRIX}, then simply select \([A]. Then type the \textbf{x-1} key, then hit \textbf{ENTER}. To convert this to fractions, hit \textbf{MATH}, then select \textbf{“Frac”}, and hit \textbf{ENTER}.
5. To calculate \(A^{-1}B\), do as in step 4, but select \([B]\) also, so that you have \([A]-1[B]\) on your screen. Hit \textbf{ENTER} to get the answer.

CASIO:
1. Hit \textbf{MENU}, and select \textbf{3:MAT}. Select the matrix, type in its size and the entries until you have all your matrices finished.
2. To calculate \(A^{-1}\), hit \textbf{MENU}, and select \textbf{1:RUN}. Hit \textbf{OPTN}, then select \textbf{F2:MAT}, then \textbf{F1:MAT}. This will bring up a “Mat” on the screen. Then you hit \textbf{ALPHA} (Red key) and \(A\) (just below) to bring up \(A\). Then hit the \textbf{x-1} key above the “)”.
3. Repeat these steps for \(A^{-1}B\).

\textbf{Section: Gaussian Elimination}

In this section we will learn a general method for finding possible solutions to a linear system of equations. The method involves systematic elimination of the unknown from each equation in turn. We will explain the method with examples.

\textbf{Example 1.} Solve the system

\[
\begin{align*}
    x_1 + x_2 + 3x_3 &= 5 \\
    2x_2 - x_3 &= 1 \\
    -3x_1 + 2x_2 + 2x_3 &= 1
\end{align*}
\]

\textbf{Solution:} Now applying the operation \(R_3 = r_3 + 3r_1\) we have the following

\[
\begin{align*}
    x_1 + x_2 + 3x_3 &= 5 \\
    2x_2 - x_3 &= 1 \\
    5x_2 + 11x_3 &= 16
\end{align*}
\]

Applying \(R_2 = 1/2 r_2\) we have

\[
\begin{align*}
    x_1 + x_2 + 3x_3 &= 5 \\
    x_2 -.5x_3 &= .5 \\
    5x_2 + 11x_3 &= 16
\end{align*}
\]

And by \(R_3 = r_3 - 5r_2\)

\[
\begin{align*}
    x_1 + x_2 + 3x_3 &= 5 \\
    x_2 - .5x_3 &= .5 \\
    13.5x_3 &= 13.5
\end{align*}
\]
Finally we the following by applying $R_3 = r_2/13.5$

\[
\begin{align*}
    x_1 + x_2 + 3x_3 &= 5 \\
    x_2 - .5x_3 &= .5 \\
    x_3 &= 1
\end{align*}
\]

We now have that $x_3 = 1$, and other unknowns can easily be found by backward substitution into second and first equations. We have the solution $(x_1, x_2, x_3) = (1, 1, 1)$. This method is called the Gaussian Elimination method.

**Augmented matrix:** Let us consider the system of equations

\[
\begin{align*}
    x_1 + x_2 + 3x_3 &= 5 \\
    2x_2 - x_3 &= 1 \\
    -3x_1 + 2x_2 + 2x_3 &= 1
\end{align*}
\]

The augmented matrix of the above system is

\[
\begin{bmatrix}
    1 & 1 & 3 & | & 5 \\
    0 & 2 & -1 & | & 1 \\
    -3 & 2 & 2 & | & 1
\end{bmatrix}
\]

**Row Echelon Form:** A matrix is in row echelon form when

1. The entry in row 1, column 1 is a 1 and 0 appears below it
2. The first nonzero entry in each row after the first row is a 1, zeros appear below it and it appears to the right of the first nonzero entry in any row above.
3. Any row that contains all zeros to the left of the vertical bar appear at the bottom.

The row echelon form of the above augmented matrix is

\[
\begin{bmatrix}
    1 & 1 & 3 & | & 5 \\
    0 & 1 & -5 & | & .5 \\
    0 & 0 & 1 & | & 1
\end{bmatrix}
\]

which exactly our Gaussian Elimination.

**Reduced Row Echelon Form (rref):** Sometimes it is advantageous to write a matrix in reduced row echelon form. In this form, row operations are used to obtain entries that are zero above as well as below the leading 1 in a row. From reduced row echelon form one can find solution of a system directly without backward substitution. The reduced row echelon form of the above row echelon form will look like the one given below:

\[
\begin{bmatrix}
    1 & 0 & 0 & | & 1 \\
    0 & 1 & 0 & | & 1 \\
    0 & 0 & 1 & | & 1
\end{bmatrix}
\]

The left hand side of the vertical bar is the identity matrix.
Finding *reduced row echelon form* by calculator:

**TI:**
1. Hit $2^{nd}$ $x^{-1}$ then edit your matrix, hit $2^{nd}$ MODE to go to the normal screen.
2. Hit $2^{nd}$ $x^{-1}$ then go to MATH, hit 8 for `rref` (and input your matrix A using again $2^{nd}$ $x^{-1}$, hit enter.
3. You will get your reduced row echelon form.

**CASIO:**
1. Go to MENU, choose MAT, use right arrow to select size and edit your matrix
2. Go to OPTION and Press F1 for row operation

**Example 2:** Find the value of $K$ so that the given system has no solution:

\[
\begin{align*}
  x_1 + x_2 + x_3 &= 6 \\ 
  2x_1 - x_2 - x_3 &= 3 \\ 
  x_1 + 2x_2 + Kx_3 &= 0
\end{align*}
\]

Solution: Find the determinant value of the coefficient matrix and set equal to zero.

\[
\begin{vmatrix}
  1 & 1 & 1 \\
  2 & -1 & -1 \\
  1 & 2 & K
\end{vmatrix} = 0
\]

Using Sarrus’s rule see that $-K + 3 - (2K - 3) = 0 \Rightarrow K = 2$. You may also use row/column operation to find two zeros and find your result from reduced system.

**Example:** Use Cramer’s Rule to explain why the system

\[
\begin{align*}
  x + 4y &= 1 \\
  2x + 8y &= 3
\end{align*}
\]

has no solution.

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**Section: Matrices and Matrix Operations**

**Matrix:** A matrix is simply a rectangular array of numbers considered as an entity. A matrix with $m$ rows and $n$ columns in the array is written as follows:

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{n2} & \cdots & a_{mn}
\end{pmatrix}
\]

and a column vector looks like $b = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{pmatrix}$ which is of order $m \times 1$
Example 1. Given $A = \begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix}$, $B = \begin{pmatrix} -5 & -2 \\ 3 & -4 \end{pmatrix}$ and $C = \begin{pmatrix} 5 & 6 \\ 5 & 10 \end{pmatrix}$. Find the following results:

a) $A + 2B - C$

b) $A + B - 0.5C$

Example 2. Solve for $x$ and $y$:

$$
\begin{bmatrix} 3 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} x - y & 2 & -2 \\ 4 & x & 6 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 2x + y & 5 \end{bmatrix}
$$

Example 3.

$$
A = \begin{bmatrix} 4 & 10 \\ -2 & -3 \\ 6 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 0 \\ 0 & 1 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix}
$$

$A$ is a matrix $3$ by $2$, $B$ is $3$ by $3$, $C$ is $3$ by $2$

$A + B = \text{Not Possible}$

$A + C = \begin{bmatrix} 4+5 & 10+0 \\ -2+(-1) & -3+3 \\ 6+4 & 9+7 \end{bmatrix} = \begin{bmatrix} 9 & 10 \\ -3 & 0 \\ 10 & 16 \end{bmatrix}$

$A - C = \begin{bmatrix} 4-5 & 10-0 \\ -2-(-1) & -3-3 \\ 6-4 & 9-7 \end{bmatrix} = \begin{bmatrix} -1 & 10 \\ -1 & -6 \\ 2 & 2 \end{bmatrix}$

Example 4.

$$
A = \begin{bmatrix} 4 & 10 \\ -2 & -3 \\ 6 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 0 \\ 0 & 1 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix}
$$

$$
D = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 3 & -4 \end{bmatrix}, \quad F = \begin{bmatrix} -1 & 2 \\ 8 & 7 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}
$$

Order of above matrices:

A is a matrix $3$ by $2$, $B$ is $3$ by $3$, $C$ is $3$ by $2$, $D$ is $3$ by $1$, $E$ is $1$ by $3$, $F$ and $G$ are $2$ by $2$ matrices.
Find the following if possible

1. \(-B,\) 
2. \(3A-2C,\) 
3. \(F+3G,\) 
4. \(2B-5C\)

Answers:

\[-B = \begin{bmatrix} -1 & -3 & -2 \\ -2 & -3 & 0 \\ 0 & -1 & -5 \end{bmatrix}, \quad 3A-2C = 3 \begin{bmatrix} 4 & 10 \\ -2 & -3 \\ 6 & 9 \end{bmatrix} - 2 \begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 30 \\ -4 & -15 \\ 10 & 13 \end{bmatrix} \]

\[F+3G = \begin{bmatrix} -1 & 2 \\ 8 & 7 \end{bmatrix} + 3 \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 17 \\ 11 & 25 \end{bmatrix} \]

\[2B-5C \text{ is not possible} \]

Group work: now try

Consider \(A = \begin{bmatrix} 2 & x \\ 6 & y \end{bmatrix},\) \(B = \begin{bmatrix} -1 & p & 2 \\ 5 & 3 & -r \end{bmatrix},\) \(C = \begin{bmatrix} t & 3 \\ s & 2 \end{bmatrix}\)

Find \(2A + 2C, 2A+B\)

**Example 6.** Given the matrix \(A = \begin{pmatrix} 2 & 2b \ 3 & 3a + 3b \end{pmatrix}.\) Find the determinant value of \(4A.\)

Answer is zero.

**Section: Matrix Multiplication**

The multiplication of two matrices \(A\) and \(B\) is a unique matrix \(AB\) which is defined for \(A\) of order \(m \times n,\) while \(B\) is of order \(n \times p.\) Note that the order of \(AB\) is \(m \times p.\)

**Example 1.** Given \(A = \begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix}, B = \begin{pmatrix} -5 & -2 \\ 3 & -4 \end{pmatrix}\) and \(C = \begin{pmatrix} 5 & 6 \\ 5 & 10 \end{pmatrix}.\) Find the following results:

a) \(BC\)  

b) \(AB\)  

c) \(BC + CA\)

**Example 2.** Find \(x\)

\[
\begin{bmatrix} -4 & -5 & 4 \\ -5 & 0 & -1 \\ 4 & -35 & -3 \end{bmatrix} \begin{bmatrix} x \\ -1 \\ 0 \end{bmatrix} = O
\]

\[
\Rightarrow \begin{bmatrix} -4x - 6 & -5x - 35 & 4x - 5 \end{bmatrix} \begin{bmatrix} x \\ -1 \\ 0 \end{bmatrix} = O
\]

\[
\Rightarrow \begin{bmatrix} 4x - 6 + 5x + 35 + 3(4x - 5) \end{bmatrix} = O
\]

\[
\Rightarrow -4x^2 - 6x + 5x + 35 + 12x - 15 = 0
\]

Now solve for \(x\) (Answer should be 4, -5/4)
Example 3. Multiply the matrices

\[
\begin{pmatrix}
a & 1-a \\
1+a & -a
\end{pmatrix}
\begin{pmatrix}
-2 & 6 & 3 \\
2 & 2 & 0
\end{pmatrix}
= \begin{pmatrix}
-2a + 2(1-a) & 6a + 2(1-a) & 3a \\
-2(1+a) - 2a & 6(1+a) - 2a & 3(1+a)
\end{pmatrix}
\]

Example 4. Let \( A = \begin{pmatrix} 2 & -1 \\ 6 & 3 \end{pmatrix}, \ B = \begin{pmatrix} -1 & 4 & 2 \\ 5 & 3 & -2 \end{pmatrix}, \ C = \begin{pmatrix} 4 & 3 \\ -5 & 2 \end{pmatrix} \). Determine the following or explain why the answer does not exist.

a) Product \( AB \). Answer: \( AB = \begin{pmatrix} -7 & 5 & 6 \\ 9 & 33 & 6 \end{pmatrix} \)

b) Sum \( 3A + 2C \). Answer: \( 3A + 2C = \begin{pmatrix} 16 & 7 \\ -3 & 12 \end{pmatrix} \)

c) Product \( BC \). Answer: Does not exist

d) Sum \( A + 3B \). Answer: Does not exist

Matrix Multiplication:
Matrix Multiplication is possible if number of column of first matrix is equal to number of row of 2nd matrix.

Example:
\[
A = \begin{bmatrix}
4 & 10 \\
-2 & -3 \\
6 & 9
\end{bmatrix}
\quad B = \begin{bmatrix}
1 & 3 & 2 \\
2 & 3 & 0 \\
0 & 1 & 5
\end{bmatrix}
\quad C = \begin{bmatrix}
5 & 0 \\
-1 & 3 \\
4 & 7
\end{bmatrix}
\]

Find the following:
\( AB \) Not possible Because \( A \) is \( 3 \times 2 \) and \( B \) is \( 3 \times 3 \)

\[
AB = \begin{bmatrix}
1(4) + 3(-2) + 2(6) & 1(10) + 3(-3) + 2(9) \\
2(4) + 3(-2) + 0(6) & 2(10) + 3(-3) + 0(9) \\
0(4) + 1(-2) + 5(6) & 0(10) + 1(-3) + 5(9)
\end{bmatrix} = \begin{bmatrix}
10 & 19 \\
2 & 11 \\
28 & 42
\end{bmatrix}
\]

Now try for \( BC \) and \( CB \).
Properties of addition and scalar multiplication:

\[(A + B) + C = A + (B + C)\]
\[A + B = B + A\]
\[A + 0 = A\]
\[A + (-A) = 0\]
\[(\alpha + \beta)A = \alpha A + \beta A\]
\[\alpha(\alpha + B) = \alpha A + \alpha B\]
\[(AB)C = A(BC)\]
\[A(B + C) = AB + BC\]
\[(A + B)C = AC + BC\]
\[AI_n = I_n A = A\] where \(I_n\) is \(n \times n\) identity matrix

**Exercise 5.** From the given matrices find \((I - A)D\), where \(I\) is the identity matrix.

\[A = \begin{pmatrix} 0.2 & 0.5 \\ 0.3 & 0.4 \end{pmatrix}, \quad D = \begin{pmatrix} 200 \\ 400 \end{pmatrix}\]

**Example 6.** Let \(A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\). Calculate \(A^2\), \(A^3\), \(A^4\), then look at the pattern and find \(A^n\), for \(n\), a positive integer.

Answer: \(A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\)

**Section 7.5 Rules for Matrix Multiplication (More examples Page # 282)**

**Example 1.** Suppose \(P\), and \(Q\) are \(n \times n\) square matrices such that \(PQ = Q^2 P\). Prove that \((PQ)^2 = Q^6 P^2\).

**Example 2.** For the following matrices verify that \(AB \neq BA\).

\[A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}\]

**Example 3.** \(A = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\), show that \(\det(3A) = 3^2 \det(A)\), find also \(\det((3A)^{-1})\)

**Example 4.** Prove that if \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\), then \(A^2 = (a + d)A - (ad - bc)I\), where \(I\) is a 2 by 2 identity matrix.
Section: The Inverse of a Matrix

For a given non-singular matrix $A$, the inverse matrix $B = A^{-1}$ exists such that $AB = BA = I$, where $I$ is an identity matrix of order same as $A$ or $B$.

A matrix $A$ is non-singular iff $\det(A) = 1 = A \neq 0$

To find inverse of a nonsingular matrix using calculator:

Step 1. Input the matrix say $A$

Step 2. Call matrix $A$ and hit $x^{-1}$ in your calculator then hit MATH and select 1 : $\Rightarrow$ Frac to get the matrix along with determinant value.

Example 1. Find the inverse of a two by two matrix by hand:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \frac{d}{ad - bc} & - \frac{b}{ad - bc} \\ - \frac{c}{ad - bc} & \frac{a}{ad - bc} \end{pmatrix}$$

Now verify that $AA^{-1} = A^{-1}A = I$

Example 2. Show that $(AB)^{-1} = B^{-1}A^{-1}$

Solution: We can consider that

$$AB(AB)^{-1} = I$$

$$A^{-1}AB(AB)^{-1} = A^{-1}I = A^{-1}, \text{ multiplying by } A^{-1}$$

$$B^{-1}IB(AB)^{-1} = B^{-1}A^{-1}, \text{ using } A^{-1}A = I \text{ and multiplying by } B^{-1}$$

$$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Example 3. Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ using $(AB)^{-1} = B^{-1}A^{-1}$

Example 4. Solve the following system of equations by matrix inverse:

$$\begin{cases} x + y - z = 12 \\ 2x - y + 2z = -3 \\ x + 2y - z = 6 \end{cases}$$

Solution: We have the following matrix system

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ -3 \\ 6 \end{pmatrix}, \text{ check for det } \begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 4$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \\ -4 & 0 & 4 \\ -5 & 1 & 3 \end{pmatrix} \begin{pmatrix} 12 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 27 \\ 4 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 27 \\ 4 \\ -6 \end{pmatrix}$$
Example 5. Show that the following system has no solution using inverse matrix method.
\[
\begin{align*}
    x + y - z &= 12 \\
    2x - y - 2z &= -3 \\
    x + 2y - z &= 6
\end{align*}
\]

Section: Determinant of order 2 by 2 and Cramer’s Rule

We have discussed about it in the previous lecture. Check your note and also you may go over your test book. Some more exercise problems are given below:

Exercise 1. Let \( A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \), \( B = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \) show that \( |AB| = |BA| \)

Exercise 2. Let \( A = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \), \( B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \), calculate \( |A^2| + |A|^2, |B^2| + |B|^2 \)

Exercise 3. Let \( A = \begin{pmatrix} .2 & .6 & .2 \\ 0 & .2 & .4 \\ .2 & 2 & 0 \end{pmatrix} \) find \( |A - 2I|^{-1} + |(A - 2I)^{-1}| \)

Exercise 4. Show that \( \begin{vmatrix} 2 & 1-a & 0 \\ 1 & 0 & 1-a \end{vmatrix} = a(1-a)(a-6) \)

Exercise 5. If \( \begin{vmatrix} 5-a & 2 & 1 \\ 2 & 1-a & 0 \\ 1 & 0 & 1-a \end{vmatrix} = 0 \) find all values of \( a \). Answer: \( a = 0, 1, 6 \)

Exercise 6. If \( A = \begin{pmatrix} 2x^2 & 3x+1 \\ 4x & x-1 \end{pmatrix} \) determine \( \frac{dA}{dx} \) Answer: \( 4x-12 \)

More examples:

Determinant:

2 by 2: \( \begin{vmatrix} 2 & 3 \\ 5 & 9 \end{vmatrix} = 18-15 = 3 \)

3 by 3: \( \begin{vmatrix} a & 2 & 1 \\ 2 & a & 3 \\ 1 & 2 & a \end{vmatrix} = a \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 1 & a \end{vmatrix} + 1 \begin{vmatrix} 2 & a \\ 1 & 2 \end{vmatrix} = a(a^2 - 6) - 2(a - 3) + 1(4 - a) \)

Another way of expansion (making two zeros by operation)
$$\begin{vmatrix} a & 2 \mathbf{1} \\ 2 & a \mathbf{3} \\ 1 & 2 & a \end{vmatrix} = \begin{vmatrix} 0 & 2-2a & 1-a^2 \\ 0 & a-4 & 3-2a \\ 1 & 2 & a \end{vmatrix} = 1 \begin{vmatrix} 2-2a & 1-a^2 \\ a-4 & 3-2a \end{vmatrix} = (1-a) \begin{vmatrix} 2 & 1+a \\ a-4 & 3-2a \end{vmatrix} = (1-a)(6-4a-(a^2-3a-4)) = (1-a)(10-a-a^2) = (a-1)(a^2+a-10)$$

Now verify the following:

1.

| 7 & 17 & 29 | 265 & 240 & 219 | 3 & 4 & 0 |
|---|---|---|---|---|---|---|
| 11 & 19 & 31 | 36 | b) 240 & 225 & 198 | 0 | c) 2 & 1 & 3 | 28 |
| 13 & 23 & 37 | 219 & 198 & 181 | 1 & -4 & 4 |

2. Prove the following:

a) \[\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ 1 & x & y+z \end{vmatrix} = (ax^2 + 2bxy + cy^2)(b^2 - ac)\]

b) \[\begin{vmatrix} 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0\]

c) \[\begin{vmatrix} a & -a & b \\ -a & 0 & -b \end{vmatrix} = a(b-3a)\]

d) \[\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)\]

e) \[\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)\]

f) \[\begin{vmatrix} 1 & 1 & 1 \\ a^2+bc & b^2+ca & c^2+ab \\ a^3 & b^3 & c^3 \end{vmatrix} = -(a-b)(b-c)(c-a)(a^2+b^2+c^2)\]

g) \[\begin{vmatrix} 1 & 1 & 1 \\ b+c & a & a \\ b & c+a & b \end{vmatrix} = 4abc\]

h) \[\begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3\]
3. Solve for all $x$, $a$, $b$, or $c$:

\[
\begin{bmatrix}
1 & 1 & 1 \\
x & a & b \\
x^2 & a^2 & b^2
\end{bmatrix} = 0
\]

\[
\begin{bmatrix}
a & b & c \\
a^2 & b^2 & c^2 \\
b + c & c + a & a + b
\end{bmatrix} = 0
\]

\[
\begin{bmatrix}
a & -a & b \\
-a & 0 & -b
\end{bmatrix} = 0
\]

Answer: a) $x = a = b$

b) $a = b = c = 0$

c) $a = b = 0$

Matrices:

4. Given $A = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$, find $A^2$

5. Given $A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 4 & 5 \end{bmatrix}$ find $BA$

6. For a given matrix $A = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$, the transpose is $A' = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$, first row becomes first column and second row becomes second column. A matrix $A$ is called orthogonal (perpendicular) if $AA' = A'A = I$. Show that the matrix

\[
A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & -2 \\ -2 & 2 & -1 \end{bmatrix}
\]

is orthogonal.

7. Given $A = \begin{bmatrix} 9 & 1 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 7 & 15 \end{bmatrix}$. If $3A + 5B + 2X = O$, where $O$ is the zero matrix, find the matrix $X$.

8. If $4 \cdot 5 + x \cdot y = 7 \cdot 3$, find $x$ and $y$

9. Suppose that $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, and $f(x) = x^2 - 5x + 6$ find $f(A)$

\[
\begin{bmatrix} 4 \\ 1 \end{bmatrix}
\]

\[
\begin{bmatrix} -4 & 8 & 4 \\ -3 & 6 & 3 \end{bmatrix}
\]

10. Given $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$, find $x$, $y$ and $z$.

11. Solve the following system using both Cramer’s rule and matrix inverse:
a) \[
\begin{align*}
2x - 3y &= 3 \\
4x - y &= 11
\end{align*}
\]

b) \[
\begin{align*}
5a - 6b + 4c &= 15 \\
7a + 4b - 3c &= 19 \\
2a + b + 6c &= 46
\end{align*}
\]

c) \[
\begin{align*}
2x + 3y + z &= 110 \\
3x + 5y + 2z &= 190 \\
3z + y + x &= 120
\end{align*}
\]

d) \[
\begin{align*}
2x + 3y - z + 12 &= 0 \\
3x + 4y + 11z &= 46 \\
5y + 4z &= 5
\end{align*}
\]

e) \[
\begin{align*}
2x - y &= 5 \\
x + 2y &= -3
\end{align*}
\]

12. Given an input matrix \( A = \begin{bmatrix} 240 & 750 \\ 1200 & 1500 \\ 720 & 450 \\ 1200 & 1500 \end{bmatrix} \) and the demand matrix \( D = \begin{bmatrix} 210 \\ 330 \end{bmatrix} \).

Find the output matrix \( X \), such that \((I - A)X = D\).

13. Find \( k \) so the system has no solution:
\[
\begin{align*}
2x - 3y &= 3 \\
kx - y &= 11
\end{align*}
\]

14. How do you determine whether the system \[
\begin{align*}
2x - 3y &= 3 \\
x - 1.5y &= 11
\end{align*}
\]
has infinitely many solutions or no solution at all?

15. How do you determine whether the system \[
\begin{align*}
2x - 3y &= 3 \\
x - 1.5y &= 1.5
\end{align*}
\]
has infinitely many solutions or no solution at all?

16. Find an equilibrium vector for the transition matrix given below:
\[
P = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.4 & 0.2 \end{bmatrix}
\]

Solution: The equilibrium vector is a row matrix \( v = [x \ y \ z] \), \( x + y + z = 1 \) such that \( vP = v \). Use matrix multiplication and find \( v \).

\[
\begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.4 & 0.2 \end{bmatrix}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x \ y \ z] \Rightarrow
\begin{align*}
0.4x + 0.3y + 0.4z &= x \\
0.4x + 0.3y + 0.4z &= y \\
0.2x + 0.4y + 0.2z &= z \\
x + y + z &= 1
\end{align*}
\]

Solve this system:
\[
\begin{align*}
x &= 0.36364 = \frac{4}{11}, \\
y &= 0.36364 = \frac{4}{11}, \\
z &= 0.27272 = \frac{3}{11}
\end{align*}
\]

The required equilibrium vector is \( v = \begin{bmatrix} \frac{4}{11} \ 4/11 \ 3/11 \end{bmatrix} \) which is a \( 1 \) by \( 3 \) matrix.