Chapter 6 Polar Coordinates

6.1 Rectangular Coordinate System and Plotting Points

The Coordinate Plane

The rectangular Cartesian coordinate system has been used throughout the book to represent curves, points in the plane. In this section we discuss another system called polar coordinates. We will also present the relation between the polar coordinate system to the Cartesian system. To represent a point in the Cartesian system by (2, 3) we move 2 units along the $x$– axis and then move 3 units along the $y$– axis. On the other hand the corresponding representation in polar system is the distance of the point from the origin is determined along with the angle formed by the line segment with $x$– axis in the positive direction. Thus the polar representation of $(2, 3)$ is $(\sqrt{13}, \arctan(3/2))$. The general representation of a point $(x, y)$ in Cartesian system to its corresponding polar form is $(r, \theta)$, where

\[
x = r \cos \theta, \quad y = r \sin \theta
\]

\[
x^2 + y^2 = r^2, \quad \tan \theta = \frac{y}{x}, \quad x \neq 0
\]

![Figure 7.1](image)

Examples

1. Find and plot the polar coordinate system of the point that has Cartesian form $(1, -1)$.

Solution We have

\[
r = \pm \sqrt{x^2 + y^2} = \pm \sqrt{2}, \quad \theta = \arctan(-1) = -\frac{\pi}{4}.
\]

The point on the polar coordinate could be either $(\sqrt{2}, -\pi/4)$ or $(-\sqrt{2}, 3\pi/4)$ or $(\sqrt{2}, 7\pi/4)$ etc.

![Figure 7.2](image)
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2. The polar representation of a point is given by \((3, \pi/4)\). Find its Cartesian form.

**Solution** We have
\[
x = r \cos \theta = 3 \cos(\pi/4) = \frac{3\sqrt{2}}{2},
\]
\[
y = r \sin \theta = 3 \sin(\pi/4) = \frac{3\sqrt{2}}{2}.
\]
The point in the Cartesian coordinate system is found as \(\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)\).

3. Find the Cartesian form of the equation given in polar form as \(r = 3\sin \theta + 3\cos \theta\)

**Solution** We have
\[
r^2 = 3r \sin \theta + 3r \cos \theta
\]
\[
x^2 + y^2 = 3x + 3y
\]
\[
\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{2}
\]
Which is a circle center at \(\left(\frac{3}{2}, \frac{3}{2}\right)\) and passing through the origin.

### 6.2 Cardioids

The heart shaped curve in the polar coordinate of the form \(r = a + b \cos \theta\) or \(r = a + b \sin \theta\) is called a cardioid. It is a member of a family of curves called limaçons (Pronounced as lim-a-sons). Following are some of those cardioids (the horizontal arrow is the polar axis)

\(r = a + b \cos \theta\)

\(a = b\) \quad \(a > b\) \quad \(a < b\)

\(r = a - b \cos \theta\)

\(a = b\) \quad \(a > b\) \quad \(a < b\)
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\[ r = a + b \sin \theta \]

\[ r = a - b \sin \theta \]

4. Sketch the graph of the cardioid \( r = 2 + 2\cos \theta \) and transfer the equation to its Cartesian form.

**Solution**

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0</th>
<th>( \pi/6 )</th>
<th>( \pi/4 )</th>
<th>( \pi/3 )</th>
<th>( \pi/2 )</th>
<th>( 2\pi/3 )</th>
<th>( 3\pi/4 )</th>
<th>( 5\pi/6 )</th>
<th>( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>4</td>
<td>( 2 + \sqrt{3} )</td>
<td>( 2 + \sqrt{2} )</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>( 2 - \sqrt{2} )</td>
<td>( 2 - \sqrt{3} )</td>
<td>0</td>
</tr>
<tr>
<td>( x )</td>
<td>4</td>
<td>3.23</td>
<td>2.41</td>
<td>1.5</td>
<td>0</td>
<td>-0.5</td>
<td>-0.41</td>
<td>-0.23</td>
<td>0</td>
</tr>
<tr>
<td>( y )</td>
<td>0</td>
<td>1.87</td>
<td>2.41</td>
<td>2.6</td>
<td>2</td>
<td>0.87</td>
<td>0.41</td>
<td>0.13</td>
<td>0</td>
</tr>
</tbody>
</table>

\( r = 2 + 2\cos \theta \) is a cardioid, which is symmetric to \( x \) axis. We only need to choose values of \( \theta \) between 0 to \( \pi \) and complete the graph using symmetry.

The Cartesian for will be

\[ r = 2 + 2\cos \theta \]

\[ r^2 = 2r + 2r \cos \theta \]

\[ x^2 + y^2 = 2\sqrt{x^2 + y^2} + 2x \]

\[ x^2 + y^2 - 2x = 2\sqrt{x^2 + y^2} \]

\[ (x^2 + y^2 - 2x)^2 = 4(x^2 + y^2) \]

\[ x^4 + y^4 + 2x^2y^2 - 4x^3 - 4xy^2 - 4y^2 = 0, \text{ after simplification} \]

5. Sketch the graph of the cardioid \( r = 2 - 2\cos \theta \) and transfer the equation to its Cartesian form.
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Solution

<table>
<thead>
<tr>
<th>θ</th>
<th>0</th>
<th>π/6</th>
<th>π/4</th>
<th>π/3</th>
<th>π/2</th>
<th>2π/3</th>
<th>3π/4</th>
<th>5π/6</th>
<th>π</th>
</tr>
</thead>
<tbody>
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<td>2−√2</td>
<td>1</td>
<td>2</td>
<td>3+√2</td>
<td>2+√3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>0</td>
<td>0.23</td>
<td>0.41</td>
<td>0.5</td>
<td>0</td>
<td>-1.5</td>
<td>-2.41</td>
<td>-3.23</td>
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</tr>
<tr>
<td>y</td>
<td>0</td>
<td>0.13</td>
<td>0.41</td>
<td>0.87</td>
<td>2</td>
<td>2.6</td>
<td>2.41</td>
<td>1.87</td>
<td>0</td>
</tr>
</tbody>
</table>

\( r = 2 - 2 \cos \theta \) is a cardioid, which is symmetric to \( x \) axis. We only need to choose values of \( \theta \) between 0 to \( \pi \) and complete the graph using symmetry.

The Cartesian for will be

\[
\begin{align*}
    r &= 2 - 2 \cos \theta \\
    r^2 &= 2r - 2r \cos \theta \\
    x^2 + y^2 &= 2\sqrt{x^2 + y^2} - 2x \\
    x^2 + y^2 + 2x &= 2\sqrt{x^2 + y^2} \\
    (x^2 + y^2 + 2x)^2 &= 4(x^2 + y^2) \\
    x^4 + y^4 + 2x^2y^2 + 4x^3 + 4xy^2 - 4y^2 &= 0, \text{ after simplification}
\end{align*}
\]

Figure 7.8