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Chapter 4 Analytic Trigonometry

4.1 Inverse Trigonometric Functions

The trigonometric functions act as an operator on the variable (angle) \( x \), resulting in an output value \( y \). Suppose this process is reversed: given a \( y \)-value, is it possible to work backward to determine the angle \( x \) that produced \( y \)?

In simplest terms, we wish to solve equations such as

\[
\sin x = \frac{1}{2}, \quad \tan x = \frac{2}{3}, \quad \sec x = 1, \quad etc.
\]

In some cases the value \( x \) can be determined “by inspection”. For example, the simple algebraic equation \( \sin x = \frac{1}{2} \) implies that \( x = \frac{\pi}{6} \) is one possible solution. However, this method does not work this easy in general: for example, by inspection what is the solution to \( \tan x = \frac{2}{3} \)?

Review of Inverse Graphs and Inverse Functions

A function \( y = f(x) \) generates a set of points \( \{(x, y) \mid y = f(x)\} \) that are plotted on a Cartesian \((x,y)\) coordinate axis system. The result is the graph of the function.

The inverse graph of a function \( y = f(x) \) is a graph of the set of points \( \{(y, x) \mid y = f(x)\} \). In simplest terms, the inverse graph of \( y = f(x) \) is a new graph in which each ordered pair \((x, y)\) of \( y = f(x) \) has its coordinates reversed to \((y, x)\). The points \((y, x)\) and \((x, y)\) are symmetrical across the line \( y = x \); this allows a simple way to sketch the inverse graph of a function \( y = f(x) \). However, in most cases the inverse graph will not be a function since it fails the vertical line test.

To ensure that the inverse graph of a function \( y = f(x) \) is itself a function, the given function must be one-to-one. A function is one-to-one if \( f(a) = f(b) \) implies that \( a = b \). Visually, a function that is one-to-one passes the horizontal line test.

To summarize, a function that passes the horizontal line test is said to be one-to-one, and if this condition is met, its inverse graph will be a function as well. If this is true, then the inverse function is denoted as \( y = f^{-1}(x) \).

The Inverse Sine Function (Arcsine)

Consider the function \( y = f(x) = \sin x \). Its graph is given in the next page:
The sine function is not one-to-one since it does not pass the horizontal line test. However, if the domain of \( f(x) = \sin x \) is restricted to the interval \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\), then it is one-to-one and its inverse graph is therefore a function:

Therefore, the \textbf{inverse sine} function (also called the \textbf{arcsine} function, written \( \arcsin(x) \)), is defined to be the inverse graph of the function \( f(x) = \sin x \) on the closed interval \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\). The domain of the inverse sine function is \(-1 \leq x \leq 1\) and the range is \(-\frac{\pi}{2} \leq f(x) \leq \frac{\pi}{2}\). The inverse sine function returns values in the 1\textsuperscript{st} quadrant (if \( x > 0 \)) or the 4\textsuperscript{th} quadrant (if \( x < 0 \)).

\textbf{Example 1:} Determine values for (a) \( \arcsin \left( -\frac{1}{2} \right) \), (b) \( \arcsin \left( \frac{\sqrt{3}}{2} \right) \), (c) \( \arcsin \left( \frac{\pi}{3} \right) \), (d) \( \arcsin(1) \) and (e) \( \arcsin(\pi) \).

\textbf{Solutions:} The results for (a), (b) and (c) are based on the normal measures of the sine function in the first and fourth quadrants. Thus, \( \arcsin \left( -\frac{1}{2} \right) = -\frac{\pi}{6} \), \( \arcsin \left( \frac{\sqrt{3}}{2} \right) = \frac{\pi}{3} \) and \( \arcsin \left( \frac{\pi}{3} \right) = \frac{\pi}{3} \). For (d), \( \arcsin(1) = \frac{\pi}{2} \) and for (e), the solution is not defined since \( \pi \) is outside the domain.

\textbf{Example 2:} Using a calculator set to degree mode, what is \( \arcsin(0.4) \)?

\textbf{Solution:} The solution to two decimal places is the angle 23.58 degrees.
**Example 3:** Using a calculator in radian mode, what is \( \arcsin\left(-\frac{1}{5}\right) \)?

**Solution:** To two decimal places, the result is 0.17 radian.

**The Inverse Cosine Function (Arccosine)**

The cosine function \( f(x) = \cos(x) \) is not one-to-one over its usual domain of the Real numbers. However, when restricted to the closed interval \( 0 \leq x \leq \pi \), the cosine function is one-to-one, and hence its inverse graph is also a function, which will be defined at the inverse cosine function (also written \( \arccos(x) \)).

The **inverse cosine** graph \( f(x) = \arccos(x) \) is defined on the domain \( -1 \leq x \leq 1 \) with a range of \( 0 \leq f(x) \leq \pi \). It returns values in the 1st quadrant (if \( x > 0 \)) or the 2nd quadrant (if \( x < 0 \)).

**Example 4:** Determine values for (a) \( \arccos\left(-\frac{1}{2}\right) \), (b) \( \arccos\left(-\frac{3}{4}\right) \), (c) \( \arccos\left(-\frac{\sqrt{2}}{2}\right) \), (d) \( \arccos(1) \) and (e) \( \arccos(\pi) \).

**Solutions:** The results for (a), (b) and (c) are based on the normal measures of the cosine function in the first and second quadrants. Thus, \( \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \), \( \arccos\left(-\frac{3}{4}\right) = \frac{\pi}{3} \), and \( \arccos\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4} \). For (d), \( \arccos(1) = 0 \) and for (e), the solution is not defined since \( \pi \) is outside the domain.

**Example 5:** What radian measure solves \( \arccos(0.215) \)?

**Solution:** The answer is 1.354 radians.

**Example 6:** If \( \sin(x) = \frac{\sqrt{3}}{2} \), find \( \cos(x) \).

**Solution:** This is identical to the question \( \cos(\arcsin\left(\frac{\sqrt{3}}{2}\right)) \). On a right triangle, the opposite measure of one of the non-right angles is 2 and the hypotenuse is 3. With these
facts, the Pythagorean Theorem allows for the adjacent leg to be solved: \( \sqrt{3^2 - 2^2} = \sqrt{5} \). Therefore, the cosine of this angle is \( \frac{\sqrt{5}}{3} \).

The Inverse Tangent Function (Arctangent)

Like the sine and cosine function, the tangent function \( f(x) = \tan(x) \) is not one-to-one unless restricted to a smaller domain, which is usually chosen to be \( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \). Thus restricted, the inverse tangent function (written \( \arctan(x) \)) is defined as the inverse graph of the tangent function.

The domain of the arctangent function is \( -\infty < x < \infty \) and its range is \( -\frac{\pi}{2} \leq f(x) \leq \frac{\pi}{2} \). It returns values in the 1st quadrant (if \( x > 0 \)) or the 4th quadrant (if \( x < 0 \)). Notably, it has two horizontal asymptotes at \( y = -\frac{\pi}{2} \) as \( x \to -\infty \), and \( y = \frac{\pi}{2} \) as \( x \to +\infty \).

Example 7: Determine the values to (a) \( \arctan(\sqrt{3}) \), (b) \( \arctan(-1) \) and (c) \( \arctan(\frac{\pi}{4}) \).

Solutions: For (a), \( \arctan(\sqrt{3}) = \frac{\pi}{3} \), for (b), \( \arctan(-1) = -\frac{\pi}{4} \) and for (c), the result is undefined, although it trends to positive infinity as \( x \to \frac{\pi}{2}^- \).

Example 8: Evaluate \( \sin(\arctan(\frac{1}{2})) \).
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Solution: The arctan of $\frac{1}{4}$ is an angle whose opposite leg is 1 and adjacent leg is 4. Therefore, the hypotenuse is $\sqrt{1^2 + 4^2} = \sqrt{17}$. Thus, the sine of this angle is the opposite divided by the hypotenuse: $\sin(\arctan(\frac{1}{4})) = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17}$.

\[
\arctan(\frac{1}{4}) \quad \tan x = \frac{1}{4} \quad \sqrt{17} \quad 1 \\
\text{angle} = \arctan(\frac{1}{4}) \quad 4 \quad \angle = \arctan(\frac{1}{4}) \quad 4 \\
1 \quad \sin(\arctan(\frac{1}{4})) = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17}
\]

Important Results

\[
\sin^{-1}(-t) = -\sin^{-1}(t), \quad \tan^{-1}(-t) = -\tan^{-1}(t), \quad \cos^{-1}(-t) = \pi - \cos^{-1}(t)
\]

Example 9: Evaluate the following expressions

a) $\sin^{-1}(1)$  
 b) $\sin^{-1}(-1)$  
 c) $\sin^{-1}(-1/2)$  
 d) $\cos^{-1}(1)$  
 e) $\cos^{-1}(-1)$  
 f) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$  
 g) $\tan^{-1}(1)$  
 h) $\tan^{-1}(-\sqrt{3})$  
 i) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

Solution:

a) $\sin^{-1}(1) = \sin^{-1}(\sin(\pi/2)) = \pi/2$

b) $\sin^{-1}(-1) = -\sin^{-1}(1) = -\pi/2$

c) $\sin^{-1}(-1/2) = -\sin^{-1}(1/2) = -\sin(\sin(\pi/6)) = -\pi/6$

d) $\cos^{-1}(1) = \cos^{-1}(\cos(0)) = 0$

e) $\cos^{-1}(-1) = \pi - \cos^{-1}(1) = \pi - 0 = \pi$

f) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \pi - \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \pi - \pi/4 = 3\pi/4$

g) $\tan^{-1}(1) = \tan^{-1}(\tan(\pi/4)) = \pi/4$

h) $\tan^{-1}(-\sqrt{3}) = -\tan^{-1}(\sqrt{3}) = \tan^{-1}(\tan(\pi/3)) = -\pi/3$

i) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = -\tan^{-1}\tan(\pi/6) = -\pi/6$

Common Errors to Avoid

Students sometimes make the erroneous assumption that the “arcsin” cancels the “sin”, for example, $\arcsin(\sin(x)) = x$, which is not necessarily true! Pay special attention to the domain appropriate to the particular function being evaluated. Consider these following examples:
Example 10: Is it true that $\arcsin(\sin\left(\frac{\pi}{10}\right)) = \frac{\pi}{10}$?

Solution: Yes, since $-\frac{\pi}{2} \leq \frac{\pi}{10} \leq \frac{\pi}{2}$.

Example 11: Is it true that $\arcsin(\sin\left(\frac{5\pi}{4}\right)) = \frac{5\pi}{4}$?

Solution: No. The argument $\frac{5\pi}{4}$ is outside the domain of arcsine. The arcsine function returns the smallest such angle corresponding to the input. Since $\frac{5\pi}{4}$ lies in the 3rd quadrant, the arcsine function will return an angle in the 4th quadrant equivalent to $\frac{5\pi}{4}$. The correct result is $\arcsin(\sin\left(\frac{5\pi}{4}\right)) = -\frac{\pi}{4}$.

Example 12a: Is it true that $\arccos(\cos\left(\frac{3\pi}{4}\right)) = \frac{3\pi}{4}$?

Solution: Yes. The argument $\frac{3\pi}{4}$ is within the domain of the arccosine function $0 \leq \frac{3\pi}{4} \leq \pi$.

Example 12b: Is it true that $\cos(\arccos\left(\frac{3\pi}{4}\right)) = \frac{3\pi}{4}$?

Solution: No. The value $\frac{3\pi}{4}$ is outside the domain $-1 \leq x \leq 1$ of the arccosine function. This expression is undefined.

Generalized Expressions

The argument can be left as an independent variable and expressions combined to derive relationships between the sine, cosine and tangent operations with the arcsine, arccosine and arctangent inverse operations.

Example 13: Simplify the expression $\cos(\arcsin(x))$.

Solution: The $\arcsin(x)$ suggests a result $y$ such that $\sin(y) = x$. Sketching a right triangle with $x$ at the opposite leg relative to angle $y$, and 1 at the adjacent leg to $y$, the Pythagorean Theorem gives the hypotenuse as having length $\sqrt{x^2+1}$. Therefore, the cosine of this angle ($y$) is the adjacent leg divided by the hypotenuse: $\cos(y) = \cos(\arcsin(x)) = \frac{1}{\sqrt{x^2+1}}$.

Example 14: Simplify the expression $\arccos(\sin\left(\frac{x}{6}\right))$.

Solution: Since $\sin\left(\frac{x}{6}\right) = \cos\left(\frac{5\pi}{6}\right)$, rewrite $\arccos(\sin\left(\frac{x}{6}\right))$ as $\arccos(\cos\left(\frac{5\pi}{6}\right))$. Since $\frac{5\pi}{6}$ is on the domain of the arccosine function, the expression reduces to $\arccos(\cos\left(\frac{5\pi}{6}\right)) = \frac{5\pi}{6}$.
Example 15: Solve the equation $2\sin^2 x + \sin x - 1 = 0$ in the interval $0 \leq x < 2\pi$.

Solution: Factor the equation:

$$2\sin^2 x + \sin x - 1 = 0$$
$$\Rightarrow (2\sin x - 1)(\sin x + 1) = 0$$

Set each factor equal to zero and solve for $x$:

$$2\sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \arcsin\left(\frac{1}{2}\right) \Rightarrow x = \frac{\pi}{6}$$
$$\sin x + 1 = 0 \Rightarrow \sin x = -1 \Rightarrow x = \arcsin(-1) \Rightarrow x = -\frac{\pi}{2}$$

Note that the arcsine function returns values in quadrants 1 and 4 only. Furthermore, other solutions may need to be inferred from the ones provided by the arcsine operation:

Since $x = \frac{\pi}{6}$ is a solution, there also exists a solution in quadrant 2 by symmetry. Therefore, $x = \frac{5\pi}{6}$ is also a solution to the equation.

The solution $x = -\frac{\pi}{2}$ provided by the arcsine operation is correct but outside the desired bounds $0 \leq x < 2\pi$. This is easily remedied by selecting $x = \frac{3\pi}{2}$ as the equivalent solution.

Therefore, the solution set to this equation is \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\}.

Example 16: Solve the equation $\tan^2 x + 4\tan x + 3 = 0$ in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Solution: Factor the equation:

$$\tan^2 x + 4\tan x + 3 = 0$$
$$\Rightarrow (\tan x + 3)(\tan x + 1) = 0$$

Each factor is set equal to zero and solved for the variable:

$$\tan x + 3 = 0 \Rightarrow \tan x = -3 \Rightarrow x = \arctan(-3) \Rightarrow x = -1.249... \text{ radians}$$
$$\tan x + 1 = 0 \Rightarrow \tan x = -1 \Rightarrow x = \arctan(-1) \Rightarrow x = -\frac{\pi}{2} \text{ radians}$$

The solution set (in radians) is \{-1.249, -\frac{\pi}{2}\}.
4.2 Trigonometric Identities

An identity is an equation that is always true for all values in its domain. For example, the equation \( a^2 - b^2 = (a + b)(a - b) \) is an identity since it is true for all \( a \) and \( b \). In a similar manner, many trigonometric expressions can be expressed in different forms, thus forming trigonometric identities.

Reciprocal Identities

From the definitions of the six trigonometric functions, the tangent, cotangent, secant and cosecant functions can all be expressed as identities involving the sine and/or cosine functions:

\[
\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x} \quad \sec x = \frac{1}{\cos x} \quad \csc x = \frac{1}{\sin x}
\]

Simple algebraic manipulations can generate more identities. As an example, since \( \tan x = \frac{\sin x}{\cos x} \), it is possible to generate a new identity by multiplying the \( \cos x \) and deriving \( \sin x = \tan x \cdot \cos x \). However, restrictions placed on the domain \( x \) are maintained throughout any further manipulations. Thus, the expression \( \sin x = \tan x \cdot \cos x \) is true as long as \( x \neq \frac{(2n-1)\pi}{2}, n \in N \), since these restrictions are in place because of the presence of the tangent function.

The Pythagorean Identities

The sine and cosine functions are defined on the unit circle and are related by the Pythagorean identity:

\[
\sin^2 x + \cos^2 x = 1
\]

With simple algebra, new corollary identities can be formed:

\[
\sin^2 x = 1 - \cos^2 x \\
\cos^2 x = 1 - \sin^2 x
\]

These identities are true for all \( x \in \Re \).

Furthermore, the Pythagorean Identity can be divided through by \( \sin^2 x \) or \( \cos^2 x \) to generate more identities:

\[
\sin^2 x + \cos^2 x = 1 \quad \text{Divide by} \quad \sin^2 x: \quad \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \quad \Rightarrow \quad 1 + \cot^2 x = \csc^2 x
\]

\[
\sin^2 x + \cos^2 x = 1 \quad \text{Divide by} \quad \cos^2 x: \quad \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \quad \Rightarrow \quad \tan^2 x + 1 = \sec^2 x
\]

These identities are true for all \( x \) for which the expressions are defined.
General Identities and Demonstrations

The purpose of employing identities is to reduce a large trigonometric expression into something smaller and easier to manipulate. The next few examples illustrate the method to demonstrate the truthfulness of each identity. Until the proof is demonstrated, the (potential) identity is called a theorem.

The general method is to use whatever identity is appropriate at any given step – this often takes practice to “recognize” the right identity. Furthermore, there may be more than one way to demonstrate the truthfulness of the identity. The only requirement that must not be violated is that each side of the identity must be handled separately. Any algebraic maneuver that assumes the equality is true cannot be used since the truthfulness of the equation has yet to be shown! In other words, you cannot use what you are trying to prove. In most cases, this removes the method of cross-multiplication from consideration. Otherwise, normal algebraic techniques are used as needed. Always work from the most complicated form first.

Example 1: Prove that \( \frac{1 + \cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{1 + \cos x}{\sin x \cos x} \)

Solution: The left side of the theorem shows two rational expressions being summed. Rewrite the left side as a sum by using a common denominator. The common denominator of \( \frac{\cos x}{\sin x} \) and \( \frac{\sin x}{\cos x} \) is their product, \( \sin x \cos x \). The numerators must be adjusted accordingly:

\[
\frac{1 + \cos x}{\sin x} \cdot \left( \frac{\cos x}{\cos x} + \frac{\sin x}{\cos x} \cdot \frac{\sin x}{\sin x} \right) = \frac{(1 + \cos x) \cos x + \sin^2 x}{\sin x \cos x}
\]

The expression on the right side can be simplified by distributing the \( \cos x \):

\[
\frac{(1 + \cos x) \cos x + \sin^2 x}{\sin x \cos x} = \frac{\cos x + \cos^2 x + \sin^2 x}{\sin x \cos x}
\]

Since \( \sin^2 x + \cos^2 x = 1 \), the numerator reduces to

\[
\frac{\cos x + \cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{\cos x + 1}{\sin x \cos x}
\]

Thus, we have shown by transitivity the following:

\[
\frac{1 + \cos x}{\sin x} + \frac{\sin x}{\cos x} \Rightarrow \frac{1 + \cos x}{\sin x} \cdot \frac{\cos x}{\cos x} + \frac{\sin x}{\cos x} \cdot \frac{\sin x}{\sin x} \Rightarrow \frac{(1 + \cos x) \cos x + \sin^2 x}{\sin x \cos x}
\]

\[
\Rightarrow \frac{\cos x + \cos^2 x + \sin^2 x}{\sin x \cos x} \Rightarrow \frac{\cos x + 1}{\sin x \cos x}
\]

Therefore, the left side of the theorem is equal to the right side. The theorem is proven; it is an identity.
Example 2: Prove that \( \frac{\sin x + \cos x}{\sec x + \csc x} = \frac{\sin x}{\sec x} \)

Solution: The left side of the theorem contains “more” expressions with which to manipulate. Convert all functions into equivalent forms in terms of sine and cosine:

\[
\frac{\sin x + \cos x}{\sec x + \csc x} = \frac{\sin x + \cos x}{\frac{1}{\cos x} + \frac{1}{\sin x}}
\]

The denominators of the two small expressions in the denominator of the main expression can be combined by summing over a common denominator:

\[
\frac{\sin x + \cos x}{\sec x + \csc x} = \frac{\sin x + \cos x}{\frac{\sin x \cos x}{\sin x \cos x} + \frac{\cos x}{\sin x \cos x}} = \frac{\sin x + \cos x}{\frac{\sin x + \cos x}{\sin x \cos x}}
\]

Reciprocating the combined expression in the main denominator, the “\( \sin x + \cos x \)” expressions cancel:

\[
\frac{\sin x + \cos x}{\frac{\sin x + \cos x}{\sin x \cos x}} = (\sin x + \cos x)\left(\frac{\sin x \cos x}{\sin x + \cos x}\right) = \sin x \cos x
\]

Since \( \cos x = \frac{1}{\sec x} \), the expression \( \sin x \cos x \) now becomes

\[
\sin x \cos x = \sin x \left(\frac{1}{\sec x}\right) = \frac{\sin x}{\sec x}
\]

Thus, the theorem is proven.

Notice that we never once manipulated the expression \( \frac{\sin x}{\sec x} \) directly in this proof. We were able to show the relationship entirely by manipulating the expressions from the left side of the original theorem.

Shift and Reflection Identities

The sine and cosine functions are identical in shape and differ only by a horizontal shift. Specifically, the sine function is the cosine function shifted \( \frac{\pi}{2} \) units to the right. Conversely, the cosine function is the sine function shifted \( \frac{\pi}{2} \) units to the left. Therefore, two useful identities can be stated:

\[
\sin x = \cos(x - \frac{\pi}{2}) \quad \quad \quad \quad \cos x = \sin(x + \frac{\pi}{2})
\]

Furthermore, the sine function shifted \( \pi \) units to the left or right results in an inverted sine function. The same is true for the cosine function. These two identities are summarized below:

\[
\sin(x \pm \pi) = -\sin x \quad \quad \quad \cos(x \pm \pi) = -\cos x
\]
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Shifting the sine and cosine functions $\frac{\pi}{2}$ units in opposite directions result in inverted cosine and sine functions:

\[
\cos(x + \frac{\pi}{2}) = -\sin x \\
\sin(x - \frac{\pi}{2}) = -\cos x
\]

Of course, the sine and cosine functions have a period of $2\pi$. Therefore, two more identities can be stated:

\[
\sin(x \pm 2\pi) = \sin x \\
\cos(x \pm 2\pi) = \cos x
\]

The tangent function has a period of $\pi$ so its shift identity is

\[
\tan(x \pm \pi) = \tan x
\]

Shift identities for the secant, cosecant and cotangent functions can be derived by converting into sine and cosine functions.

A vertical reflection of any function can be determined by replacing the argument $x$ with $-x$. The result is a graph that is reflected across the $y$-axis. The cosine function is already symmetrical to the $y$-axis so it will remain unchanged. The sine and tangent functions, both symmetrical to the origin, will invert across the $x$-axis. Thus, the following reflection identities can be stated:

\[
\cos(-x) = \cos x \\
\sin(-x) = -\sin x \\
\tan(-x) = -\tan x
\]

For now, the “proofs” of these identities are done visually. In the next section, a method will be developed to analytically prove all such shift and reflection identities.

**Example 3:** Prove that $\sin(\frac{\pi}{2} - x) = \cos x$.

**Solution:** Factor the negative within the argument of the sine function:

\[
\sin(\frac{\pi}{2} - x) = \sin(-(x - \frac{\pi}{2}))
\]

The leading negative within the argument can be “moved” to in front of the sine function:

\[
\sin(-(x - \frac{\pi}{2})) = -\sin(x - \frac{\pi}{2})
\]

Since $\sin(x - \frac{\pi}{2}) = -\cos x$, then $-\sin(x - \frac{\pi}{2}) = -(-\cos x) = \cos x$. 
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4.3 Sum and Difference Formulas

The sum and difference forms of the sine and cosine functions are the following four forms:

\[ \sin(x + y), \cos(x + y), \sin(x - y) \text{ and } \cos(x - y). \]

The trigonometric function can be viewed as operators on the variable(s) \( x \) and \( y \), but they are not linear operators. The function/operators cannot be distributed across addition and subtraction. Hence, statements like \( \sin(x + y) = \sin x + \sin y \) are false. Unfortunately these are common mistakes students make when considering the sum and difference forms of the trigonometric functions.

In this section we will present a geometric proof for all these four forms.

We prove that

\[ \sin(x + y) = \sin x \cos y + \cos x \sin y \]

Suppose that \( OX \) be the initial side and the terminal side \( OY \) makes an angle \( x \) with initial side and the terminal side \( OZ \) makes angles \( y \) with the side \( OY \). Take a point \( P \) on the side \( OZ \) and make perpendicular lines \( PQ \) and \( PS \) on \( OX \) and \( OY \) respectively. Further we make perpendicular line \( ST \) and \( SR \) according to the adjacent diagram. Now observe that the angle \( TPS \) is equal to the angle \( x \). The angle \( POR \) is equal to \( x + y \).

\[
\sin(x + y) = \frac{PQ}{OP} = \frac{PT + TQ}{OP} = \frac{PT}{OP} + \frac{TQ}{OP} = \frac{PT}{OP} + \frac{SR}{OP} = \frac{PT}{OP} + \frac{PS}{OP} \frac{SR}{OS} \frac{OS}{OP} = \cos x \sin y + \sin x \cos y
\]

And

\[
\sin(x - y) = \sin x \cos(-y) + \cos x \sin(-y) = \sin x \cos y - \cos x \sin y, \quad \sin(-x) = -\sin x, \cos(-x) = \cos x
\]

Now we have to prove the result for cosine:

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\[
\cos(x + y) = \sin \left( \frac{\pi}{2} - (x + y) \right) = \sin \left( \left( \frac{\pi}{2} - x \right) - y \right) = \sin \left( \frac{\pi}{2} - x \right) \cos(-y) + \cos \left( \frac{\pi}{2} - x \right) \sin(-y) \\
= \cos x \cos y - \sin x \sin y, \quad \sin(-x) = -\sin x, \quad \sin \left( \frac{\pi}{2} - x \right) = \cos x, \quad \cos(-x) = \cos x
\]

Also \( \cos(x - y) = \cos x \cos(-y) - \sin x \sin(-y) = \cos x \cos y + \sin x \sin y \)

We now present the algebraic proof.

The sum and difference forms can be derived from an analysis of the angles drawn on a unit circle, and the familiar distance formula. We consider the case \( \cos(x - y) \) first, from which the other three forms can easily be derived as corollaries.

**Proof of** \( \cos(x - y) = \cos x \cos y + \sin x \sin y \)

Consider the unit circle in the first quadrant (without loss of generality). Let ray \( r_x \) be drawn with angle \( x \), and \( r_y \) be drawn with angle \( y \), and let \( x > y \). Therefore, ray \( r_x \) intersects the circle at point \((\cos x, \sin x)\) and ray \( r_y \) intersects the circle at point \((\cos y, \sin y)\).

Rigidly rotate the circle clockwise through an angle of \( y \), so that the original ray \( r_y \) sits along the positive \( x \)-axis while original ray \( r_x \) now has an angle of elevation \( x - y \) and therefore intersects the unit circle at the point \((\cos(x - y), \sin(x - y))\).

The core of the proof is to show that the distance between points \((\cos x, \sin x)\) and \((\cos y, \sin y)\) is the same as the distance from point \((\cos(x - y), \sin(x - y))\) to the point \((1,0)\). See the following diagram:

![Diagram](https://via.placeholder.com/150)

*Figure.* Diagram showing rays \( r_x \) and \( r_y \) (left), and the rotation through an angle \( y \) (right).

The distance between \((\cos x, \sin x)\) and \((\cos y, \sin y)\) is found by the distance formula:
The Pythagorean identity \( \sin^2 x + \cos^2 x = 1 \) was used twice in the second-to-third step.

The distance between \((\cos(x - y), \sin(x - y))\) and \((1,0)\) is

\[
D((\cos(x - y), \sin(x - y)), (1,0)) = \sqrt{(\cos(x - y) - 1)^2 + (\sin(x - y) - 0)^2} = \sqrt{\cos^2(x - y) - 2\cos(x - y) + 1 + \sin^2(x - y)} = \sqrt{2 - 2\cos(x - y)}
\]

The Pythagorean identity \( \sin^2(x - y) + \cos^2(x - y) = 1 \) was used in the second-to-third step.

Since the two distances are equal, relate them by equality:

\[
\sqrt{2 - 2\cos(x - y)} = \sqrt{2 - 2\cos x \cos y - 2\sin x \sin y}
\]

Squaring both sides removes the radicals. The constants cancel, and the above equation algebraically reduces to:

\[
\cos(x - y) = \cos x \cos y + \sin x \sin y \quad (A)
\]

Therefore, the difference formula for the cosine function is proved.

**Proofs of the Remaining Forms**

The other three forms can be derived using the difference form of the cosine function \((A)\) above. For example, using the shift identities \(\cos(x - \frac{x}{2}) = \sin x\) and \(\sin(x - \frac{x}{2}) = -\cos x\), we can make the substitution \(x - \frac{x}{2}\) for \(x\) into the above form \((A)\) to get

\[
\cos(x - \frac{x}{2} - y) = \cos(x - \frac{x}{2}) \cos y + \sin(x - \frac{x}{2}) \sin y \quad (B)
\]

The left side of \((B)\) can be re-arranged as \(\cos(x - \frac{x}{2} - y) = \cos(x - y - \frac{x}{2})\), which equals \(\sin(x - y)\). On the right side of \((B)\), we use the substitutions \(\cos(x - \frac{x}{2}) = \sin x\) and \(\sin(x - \frac{x}{2}) = -\cos x\). Therefore, equation \((B)\) can be written as

\[
\sin(x - y) = \sin x \cos y - \cos x \sin y
\]

This proves the difference formula for the sine function.
The sum forms can be proven using the symmetry identities \( \sin(-x) = -\sin x \) and \( \cos(-x) = \cos x \). Therefore, the term \( y \) is substituted with \( -y \) in equation (A) to get

\[
\cos(x - (-y)) = \cos x \cos(-y) + \sin x \sin(-y)
\]
\[
\Rightarrow \cos(x + y) = \cos x \cos y - \sin x \sin y
\]

This proves the sum form for the cosine function.

The same substitution is made in equation (B):

\[
\sin(x - (-y)) = \sin x \cos(-y) - \cos x \sin(-y)
\]
\[
\Rightarrow \sin(x + y) = \sin x \cos y + \cos x \sin y
\]

This proves the sum form for the sine function.

The Sum and Difference Identities for the Sine and Cosine Functions

| \sin(x + y) = \sin x \cos y + \cos x \sin y |
| \cos(x + y) = \cos x \cos y - \sin x \sin y |
| \sin(x - y) = \sin x \cos y - \cos x \sin y |
| \cos(x - y) = \cos x \cos y + \sin x \sin y |

The following examples illustrate some of the ways these formulas can be used:

**Example 1**: Calculate \( \sin(\frac{\pi}{12}) \) exactly.

**Solution**: The angle \( \frac{\pi}{12} \) is the difference of angles \( \frac{\pi}{4} \) and \( \frac{\pi}{3} \): \( \frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4} \). Therefore,

\[
\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}
\]

**Example 2**: Calculate \( \cos(105^\circ) \) exactly.

**Solution**: 105 degrees can be written as the sum of 60 and 45 degrees. Therefore,

\[
\cos(105^\circ) = \cos(60^\circ + 45^\circ) = \cos(60^\circ)\cos(45^\circ) - \sin(60^\circ)\sin(45^\circ) = \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}
\]

**Example 3**: Prove that \( \cos(x + \pi) = -\cos x \).

**Solution**: Use the sum form of the cosine function:

\[
\cos(x + \pi) = \cos x \cos \pi - \sin x \sin \pi = \cos x(-1) + \sin x(0) = -\cos x
\]
Example 4: Evaluate $\tan\left(\frac{10\pi}{12}\right)$ exactly.

Solution: The angle $\frac{10\pi}{12} - \frac{5\pi}{6}$ is the sum of angles $\frac{\pi}{3}$ and $\frac{\pi}{2}$. Therefore,

$$\sin\left(\frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = \frac{\pi}{2} \cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

Alternate proof: $\sin\left(\frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

**Sum and Difference Angle Formulas For Tangent**

The sum and difference formulas for the tangent function can be derived using the sum and difference forms for the sine and cosine function.

For $\tan(x + y)$, we rewrite this in terms of sine and cosine:

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)}$$

The quotient is rewritten using the sum forms:

$$\frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

Multiply the numerator and denominator by the expression $\frac{1}{\cos x \cos y}$ and simplify:

$$\frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} = \frac{(\frac{1}{\cos x \cos y})(\sin x \cos y + \cos x \sin y)}{(\frac{1}{\cos x \cos y})(\cos x \cos y - \sin x \sin y)} = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Therefore, the sum formula for the tangent function is:

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

The difference formula is found by replacing $y$ with $-y$:
The two forms can be summarized together as

\[
\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}
\]

**Example 5:** Evaluate \(\tan\left(\frac{\pi}{12}\right)\) exactly.

**Solution:** The angle \(\frac{\pi}{12}\) is the sum of angles \(\frac{\pi}{3}\) and \(\frac{\pi}{4}\). Therefore,

\[
\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \frac{\tan \frac{\pi}{3} + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{3} \tan \frac{\pi}{4}} = \frac{\sqrt{3} + 1}{1 - \sqrt{3} \cdot 1} = -\frac{(\sqrt{3} + 1)^2}{2}
\]

In the final step, the denominator was rationalized by multiplication of the conjugate.
4 Analytic Trigonometry

4.4 Double and Half-Angle Formulas

A natural extension of the sum and difference formulas (section 4.3) are the double-angle formulas. For example, the expression \(\sin(2x)\) is a double-angle expression, since the argument “\(2x\)” is interpreted as “double the angle \(x\)”.

Using the sum formula for \(\sin x\), the expression \(\sin(2x)\) can be rewritten as:

\[
\sin(2x) = \sin(x + x) = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x
\]

This is an extremely useful identity known as the double-angle identity for sine.

The double-angle identity for the cosine function is derived similarly:

\[
\cos(2x) = \cos(x + x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x
\]

Using the Pythagorean identities (section 4.2) for \(\sin^2 x\) or \(\cos^2 x\), the double-angle identity for cosine can be rewritten in two corollary forms:

\[
\cos(2x) = (1 - \sin^2 x) - \sin^2 x = 1 - 2 \sin^2 x
\]

\[
\cos(2x) = \cos^2 x - (1 - \cos^2 x) = 2 \cos^2 x - 1
\]

For the tangent function, its double-angle identity is

\[
\tan(2x) = \tan(x + x) = \frac{\tan x + \tan x}{1 - \tan x \tan x} = \frac{2 \tan x}{1 - \tan^2 x}
\]

All forms can be summarized as the double angle identities: These formulas are very useful and can be incorporated into many of the common identity proofs that are encountered.

The Double-Angle Identities

\[
\begin{align*}
\sin(2x) & = 2 \sin x \cos x \\
\cos(2x) & = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - \sin^2 x \\
\tan(2x) & = \frac{2 \tan x}{1 - \tan^2 x}
\end{align*}
\]

Example 1: Prove that \((\sin x + \cos x)^2 = 1 + \sin(2x)\)

Solution: Expand the left side by multiplication:

\[
(\sin x + \cos x)^2 = \sin^2 x + 2 \sin x \cos x + \cos^2 x
\]

The first and third terms sum to 1 by the Pythagorean Identity. The middle term is \(\sin(2x)\). Therefore, the identity is proven.
Example 2: Let angle \( x \) be given in the following diagram. Calculate \( \sin(2x) \) and \( \cos(2x) \).

\[ \text{Solution:} \] The length of the hypotenuse is found by the Pythagorean formula:

\[ \text{hyp} = \sqrt{4^2 + 7^2} = \sqrt{65} \]

Therefore, for \( \sin(2x) \), we use its double angle formula:

\[ \sin(2x) = 2 \sin x \cos x = 2\left(\frac{4}{\sqrt{65}}\right)\left(\frac{7}{\sqrt{65}}\right) = \frac{56}{65} \]

For \( \cos(2x) \), any one of the three double-angle formulas may be used:

\[ \cos(2x) = \cos^2 x - \sin^2 x = \left(\frac{7}{\sqrt{65}}\right)^2 - \left(\frac{4}{\sqrt{65}}\right)^2 = \frac{49}{65} - \frac{16}{65} = \frac{33}{65} \]

The Pythagorean identity could also have been used to determine \( \cos(2x) \):

\[ \cos^2(2x) = 1 - \sin^2(2x) = 1 - \left(\frac{56}{65}\right)^2 = 1 - \frac{3136}{4225} = \frac{1089}{4225} \]

\[ \Rightarrow \cos(2x) = \sqrt{\frac{1089}{4225}} = \frac{33}{65} \]

It is interesting to note that the numbers 33, 56 and 65 form an integer solution to the Pythagorean formula: \( 33^2 + 56^2 = 65^2 \). In fact, the double-angle formulas for sine and cosine always generate a Pythagorean “triple” of integer solutions, as the next example illustrates:

Example 3: Suppose a right triangle has angle acute angle \( x \) as one of its interior angles. Let the opposite leg be \( a \) and the adjacent leg be \( b \). Determine \( \sin(2x) \) and \( \cos(2x) \) and show that the numerators and the denominator form an integer Pythagorean triple.

\[ \text{Solution:} \] The hypotenuse is \( \text{hyp} = \sqrt{a^2 + b^2} \). For \( \sin(2x) \), we get

\[ \sin(2x) = 2 \sin x \cos x = 2\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{b}{\sqrt{a^2+b^2}}\right) = \frac{2ab}{a^2+b^2} \]

Similarly,

\[ \cos(2x) = \cos^2 x - \sin^2 x = \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 - \left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 = \frac{b^2-a^2}{a^2+b^2} \]

The expressions \( 2ab \), \( b^2 - a^2 \) and \( b^2 + a^2 \) are integers and form an integer solution to the Pythagorean formula:
The equality is true. This is an interesting way to generate integer solutions to the Pythagorean formula, and shows also that the number of possible integer solutions to the Pythagorean formula is infinite.

**Example 4:** If \( \sin x = \frac{2}{5} \) and \( x \) is in the first quadrant, find \( \sec(2x) \).

**Solution:** We will need the cosine of \( x \). Since the opposite leg is 2 and the hypotenuse 5, the adjacent leg is \( \text{adj} = \sqrt{5^2 - 2^2} = \sqrt{21} \). Therefore, \( \cos x = \frac{\sqrt{21}}{5} \). To determine \( \sec(2x) \), rewrite it as

\[
\sec(2x) = \frac{1}{\cos(2x)}
\]

The double-angle formula for cosine gives

\[
\cos(2x) = \cos^2 x - \sin^2 x = \left(\frac{\sqrt{21}}{5}\right)^2 - \left(\frac{2}{5}\right)^2 = \frac{17}{25}.
\]

Therefore, \( \sec(2x) = \frac{25}{17} \).

**The Half-Angle Formulas**

Using the double-angle formulas for sine and cosine, we can derive similar formulas for the half-angle of sine and cosine. The usual approach is to start with the double-angle formula for cosine, entirely in terms of cosine. Let \( v \) be a temporary dummy variable:

\[
\cos(2v) = 2\cos^2 v - 1
\]

Let \( v = \frac{x}{2} \), so therefore \( 2v = x \). The above formula becomes

\[
\cos x = 2\cos^2 \left(\frac{x}{2}\right) - 1
\]

Then algebraically solve for the term \( \cos\left(\frac{x}{2}\right) \):

\[
\cos x = 2\cos^2 \left(\frac{x}{2}\right) - 1 \quad \Rightarrow \quad \cos\left(\frac{x}{2}\right) = \frac{\cos x + 1}{2} \quad \Rightarrow \quad \cos\left(\frac{x}{2}\right) = \pm \sqrt{\frac{1 + \cos x}{2}}
\]

This establishes the half-angle formula for cosine. The sign is selected depending on the quadrant in which the angle \( x \) is located. To determine the half-angle formula for sine, use the Pythagorean identity \( \sin^2 \left(\frac{x}{2}\right) = 1 - \cos^2 \left(\frac{x}{2}\right) \):

\[
\sin \left(\frac{x}{2}\right) = \sqrt{1 - \left(\frac{\cos x + 1}{2}\right)^2} = \sqrt{1 - \left(\frac{1 + \cos x}{2}\right)} = \sqrt{\frac{2}{2} - \left(\frac{1 + \cos x}{2}\right)} = \pm \sqrt{\frac{1 - \cos x}{2}}
\]
Be careful to note that the two formulas both contain \( \cos x \) and they look quite similar. The half-angle formula for tangent is derived by the following steps:

\[
\tan\left(\frac{x}{2}\right) = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \sqrt{\frac{1-\cos x}{2 \cos x}} = \sqrt{\frac{1-\cos x}{1+\cos x}}
\]

This is one common way to write the half-angle formula for tangent. Another way to write the formula is to rationalize the denominator by multiplying by the conjugate:

\[
\tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{1+\cos x} \cdot \frac{(1-\cos x)}{(1-\cos x)} = \sqrt{\frac{(1-\cos x)^2}{1-\cos^2 x}} = \sqrt{\frac{1-\cos x}{\sin^2 x}} = \frac{1-\cos x}{\sin x}
\]

The half-angle formulas are summarized below:

**The Half-Angle Formulas**

\[
\begin{align*}
\sin\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{1-\cos x}{2}} \\
\cos\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{1+\cos x}{2}} \\
\tan\left(\frac{x}{2}\right) &= \frac{1-\cos x}{\sin x}
\end{align*}
\]

**Example 5:** Let angle \( x \) be given in the following diagram. Calculate \( \sin\left(\frac{x}{2}\right) \) and \( \cos\left(\frac{x}{2}\right) \).

![Diagram](image)

**Solution:** The hypotenuse is \( \sqrt{65} \). Therefore,

\[
\sin\left(\frac{x}{2}\right) = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1-\cos \frac{7}{2}}{2}} = \sqrt{\frac{65-7}{2 \times 65}} = \sqrt{\frac{65-7}{65 \times 130}}
\]

In the final step, the internal fraction was rationalized by multiplying top and bottom by \( \sqrt{65} \).

The value for \( \cos\left(\frac{x}{2}\right) \) is found via a similar calculation:

\[
\cos\left(\frac{x}{2}\right) = \sqrt{\frac{1+\cos x}{2}} = \sqrt{\frac{1+\cos \frac{7}{2}}{2}} = \sqrt{\frac{65+7}{2 \times 65}} = \sqrt{\frac{65+7}{65 \times 130}}
\]
4 Analytic Trigonometry

4.5 Product-to-Sum and Sum-to-Product Formulas

Occasionally it is desirable to convert the product of unlike sine and/or cosine terms into the sum or difference of two separate sine or cosine terms, and occasionally the reverse is true. In any case, the product-to-sum and sum-to-product identities allow us to make these conversions. These formulas are especially useful in integral calculus.

The Product-to-Sum Identities

The product-to-sum identities arise from a simple algebraic manipulation of the sum and difference formulas for the sine and cosine function. For example, we write both the sum and difference formulas for the cosine function:

\[
\cos(x + y) = \cos x \cos y - \sin x \sin y
\]
\[
\cos(x - y) = \cos x \cos y + \sin x \sin y
\]

Subtracted, the \(\cos x \cos y\) terms cancel and we get:

\[
\cos(x - y) - \cos(x + y) = 2\sin x \sin y
\]

Divide out the 2 and we have one such identity:

\[
\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]
\]

Summed, the \(\sin x \sin y\) terms cancel, and diving by 2, we get another similar identity:

\[
\cos x \cos y = \frac{1}{2}[\cos(x - y) + \cos(x + y)]
\]

The process is repeated with the sum and difference formulas for the sine function, and two more identities are derived in a similar manner. The set of product-to-sum identities are given in the next page.

The Product-to-Sum Identities

\[
\begin{align*}
\sin x \sin y &= \frac{1}{2}[\cos(x - y) - \cos(x + y)] \\
\cos x \cos y &= \frac{1}{2}[\cos(x - y) + \cos(x + y)] \\
\sin x \cos y &= \frac{1}{2}[\sin(x + y) + \sin(x - y)] \\
\cos x \sin y &= \frac{1}{2}[\sin(x + y) - \sin(x - y)]
\end{align*}
\]

Consider the following example:

**Example 1:** Write \(\sin x \sin 3x\) as the sum of two individual trigonometric terms.

**Solution:** The identity \(\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]\) is used here. We get the following:

\[
\sin x \sin 3x = \frac{1}{2}[\cos(x - 3x) - \cos(x + 3x)] = \frac{1}{2}[\cos(-2x) - \cos(4x)]
\]
Recall that the cosine absorbs negatives, so that the final result is
\[ \sin x \sin 3x = \frac{1}{2} [\cos(2x) - \cos(4x)] \]

**The Sum-to-Product Identities**

These identities are proven by performing the product-to-sum identities in reverse. Consider the identity \( \sin u \cos v = \frac{1}{2} [\sin(u + v) + \sin(u - v)] \). Multiply the 2 to remove the fraction, and let \( x = u + v \) and \( y = u - v \). Solving for \( u \) and \( v \) in terms of \( x \) and \( y \) yields the following: \( u = \frac{x+y}{2} \) and \( v = \frac{x-y}{2} \). Thus, the equation becomes:

\[
\sin u \cos v = \frac{1}{2} [\sin(u + v) + \sin(u - v)] \\
\Rightarrow 2 \sin \left(\frac{x+y}{2}\right) \cos \left(\frac{x-y}{2}\right) = \sin x + \sin y
\]

The other three identities are solved in a similar manner. The four sum-to-product identities are:

**The Sum-to-Product Identities**

\[
\begin{align*}
\sin x + \sin y &= 2 \sin \left(\frac{x+y}{2}\right) \cos \left(\frac{x-y}{2}\right) \\
\sin x - \sin y &= 2 \cos \left(\frac{x+y}{2}\right) \sin \left(\frac{x-y}{2}\right) \\
\cos x + \cos y &= 2 \cos \left(\frac{x+y}{2}\right) \cos \left(\frac{x-y}{2}\right) \\
\cos y - \cos x &= 2 \sin \left(\frac{x+y}{2}\right) \sin \left(\frac{x-y}{2}\right)
\end{align*}
\]

**Example 2:** Rewrite the sum of \( \cos 2x + \cos 3x \) as the product of two trigonometric functions.

**Solution:** The identity produces the following:

\[
\cos 2x + \cos 3x = 2 \cos \left(\frac{2x+3x}{2}\right) \cos \left(\frac{2x-3x}{2}\right) = 2 \cos(\frac{5}{2}x) \cos(-\frac{1}{2}x) = 2 \cos(\frac{5}{2}x) \cos(\frac{1}{2}x)
\]

By applying the appropriate product-to-sum identity, the latter result can easily be rewritten into its sum form.

These formulas can be used to prove various identities:

**Example 3:** Prove the identity: \( \sin 4x + \sin 6x = \cot x (\cos 4x - \cos 6x) \).
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Solution: The appropriate sum-to-product identities are used on both sides of the equation:

$$\sin 4x + \sin 6x = \cot x (\cos 4x - \cos 6x)$$

$$\Rightarrow 2 \sin \left( \frac{4x + 6x}{2} \right) \cos \left( \frac{4x - 6x}{2} \right) = \cot x \left( 2 \sin \left( \frac{6x + 4x}{2} \right) \sin \left( \frac{6x - 4x}{2} \right) \right)$$

$$\Rightarrow 2 \sin (5x) \cos (-x) = \cot x (2 \sin (5x) \sin (x))$$

$$\Rightarrow 2 \sin (5x) \cos (-x) = \left( \frac{\cos x}{\sin x} \right) (2 \sin (5x) \sin (x))$$

$$\Rightarrow 2 \sin (5x) \cos (x) = 2 \sin (5x) \cos (x)$$

Carefully follow each step and notice the cancellation taking place as well as the absorption of negative signs by the cosine function.

Example 4: Prove the identity: $\tan \left( \frac{p + q}{2} \right) = \frac{\sin p + \sin q}{\cos p + \cos q}$

Solution: It’s probably easiest to work the right side of this equation first:

$$\frac{\sin p + \sin q}{\cos p + \cos q} = \frac{2 \sin \left( \frac{p + q}{2} \right) \cos \left( \frac{p - q}{2} \right)}{2 \cos \left( \frac{p + q}{2} \right) \cos \left( \frac{p - q}{2} \right)}$$

The $\cos \left( \frac{p - q}{2} \right)$ factors cancel as well as the 2s. The identity is thus proven:

$$\frac{\sin p + \sin q}{\cos p + \cos q} = \frac{2 \sin \left( \frac{p + q}{2} \right) \cos \left( \frac{p - q}{2} \right)}{2 \cos \left( \frac{p + q}{2} \right) \cos \left( \frac{p - q}{2} \right)} = \frac{\sin \left( \frac{p + q}{2} \right)}{\cos \left( \frac{p + q}{2} \right)} = \tan \left( \frac{p + q}{2} \right)$$

In the next section we will see an example how these formulas can be used to solve a trigonometric equation.

4.6 Solution of Trigonometric Equations

An equation is that involves one or more of the various trigonometric functions is called a trigonometric equation. The variable is in the argument position of the trigonometric functions, and normally polynomial and exponential terms are not included. For example,
are all considered trigonometric equations, while
\[ x^2 + 2x - \cos x = 0 \]
is not considered a trigonometric equation since it involves (in this case) polynomial terms. Equations of this last type are usually impossible to solve by purely algebraic means. We will not consider these latter forms in this section.

**Linear Trigonometric Equations**

A linear trigonometric equation involves just one trigonometric function, usually itself a linear term. In these cases, solutions are easily found by the usual rules of algebra, and the correct use of the inverse trigonometric function at the appropriate time. Bear in mind all solutions will be in radians, and that you will be responsible for locating all solutions of the equation taking into account symmetries.

**Example 1:** Solve for \( x \): \( 2\sin x - 1 = 0 \). State the solution sets as follows: (a) all solutions, (b) solutions within \( 0 \leq x < 2\pi \).

**Solution:** Adding the 1 and dividing the 2 results in this equivalent equation:
\[ \sin x = \frac{1}{2} \]
Now is the correct time to employ the inverse sine function as an operator:
\[ \sin^{-1}(\sin x) = \sin^{-1}(\frac{1}{2}) \]
\[ \Rightarrow x = \sin^{-1}(\frac{1}{2}) \]
From our knowledge of the inverse sine function, the primary solution is \( x = \frac{\pi}{6} \). However, there is another solution. Recall the sine is positive in quadrants I and II; by symmetry, the secondary solution is \( x = \frac{5\pi}{6} \).

Thus, for (a), the set of all solutions is found by adding integer multiples of \( 2\pi \) to each solution:
\[ \{ \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi \}, n \in Q \]
where set \( Q \) represents the integers. For (b), the restriction \( 0 \leq x < 2\pi \) simply allows us to avoid listing all (infinitely many) solutions of this equation. The solution set is therefore just the set \( \{ \frac{\pi}{6}, \frac{5\pi}{6} \} \).

**Example 2:** Solve for \( x \): \( 6\cos(2x + 1) + 2 = 5 \). State the solution sets as follows: (a) all solutions, (b) solutions within \( 0 \leq x < 2\pi \).

**Solution:** Isolate the cosine term:
\[ \cos(2x + 1) = \frac{1}{2} \]
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Now we use the inverse cosine as an operator:

\[
\cos^{-1}(\cos(2x+1)) = \cos^{-1}\left(\frac{1}{2}\right)
\]

\[
\Rightarrow 2x + 1 = \cos^{-1}\left(\frac{1}{2}\right)
\]

The value for \(\cos^{-1}\left(\frac{1}{2}\right)\) is \(\frac{\pi}{3}\) (primary, quadrant I), and by symmetry, \(\frac{5\pi}{3}\) (secondary, quadrant IV). We now have two equations:

\[
2x + 1 = \frac{\pi}{3} + 2n\pi \quad \& \quad 2x + 1 = \frac{5\pi}{3} + 2n\pi
\]

Both solve the same way, yielding

\[
x = \frac{\pi - 3}{6} + n\pi \quad \& \quad x = \frac{5\pi - 3}{6} + n\pi
\]

Thus, to answer (a), the set of all solutions is

\[
\left\{ \frac{\pi - 3}{6} + n\pi, \frac{5\pi - 3}{6} + n\pi \right\}
\]

To find the solution restricted to \(0 \leq x \leq 2\pi\), we need to evaluate for values of \(n\). When \(n = 0\), we get the two principal solutions \(x = \frac{\pi - 3}{6}\) and \(x = \frac{5\pi - 3}{6}\); these two solutions lie in quadrants I and II, respectively. When \(n = 1\), we generate two more solutions \(x = \frac{7\pi - 3}{6}\) and \(x = \frac{11\pi - 3}{6}\), which lie in quadrants III and IV, respectively. When \(n = 2\), we generate two more solutions, but they place us back to the original solutions in quadrants I and II. Therefore, to answer part (b), the solution set of this equation when restricted to \(0 \leq x < 2\pi\) is

\[
\left\{ \frac{\pi - 3}{6}, \frac{5\pi - 3}{6}, \frac{7\pi - 3}{6}, \frac{11\pi - 3}{6} \right\}
\]

Note that there was no need to evaluate for \(n = -1\) since doing so generates solutions redundantly.

**Example 3:** Solve for \(x\):

\[2\cos{x}\sin{x} - \sin{x} = 0\]

Give all solutions in the interval \(0 \leq x < 2\pi\).

**Solution:** In this case we can factor \(\sin{x}\):

\[2\cos{x}\sin{x} - \sin{x} = 0\]

\[\Rightarrow \sin{x}(2\cos{x} - 1) = 0\]

Each factor can be solved for 0 separately. Setting \(\sin{x} = 0\) yields the solutions \(x = 0\) and \(x = \pi\). Setting \(2\cos{x} - 1 = 0\) yields \(\cos{x} = \frac{1}{2}\), which in turn yields solutions \(x = \frac{\pi}{3}\) and \(x = \frac{5\pi}{3}\). Therefore, the solution set to this function is \(\left\{0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}\).

**Comment:** the restriction \(0 \leq x < 2\pi\) is common in trigonometric equations since it requests only those solutions in the first four quadrants, and no more (i.e. no
redundancies). Note that this restriction includes 0 but excludes $2\pi$ since at this latter value, we are “repeating” the unit circle once again. Thus, in the previous example, there was no need to state the solution $x = 2\pi$ since it is redundant to the solution $x = 0$.

**Comment**: In the preceding example, do NOT divide by $\sin x$. In doing so, you remove all solutions that the $\sin x$ factor provides, and you potentially divide by zero, which is never a legal arithmetic maneuver.

**Non-linear Trigonometric Equations**

If a trigonometric term is raised to a power other than one, it is considered non-linear. The usual rules of algebra still apply, and we solve these forms using whatever methods are appropriate.

**Example 4**: Solve for $x$ in the interval $0 \leq x < 2\pi$: $\sin^2 x = \frac{3}{4}$.

**Solution**: The sine term is squared, so we take the radical, including both positive and negative solutions:

\[
\begin{align*}
\sin^2 x &= \frac{3}{4} \\
\Rightarrow \sqrt{\sin^2 x} &= \pm \sqrt{\frac{3}{4}} \\
\Rightarrow \sin x &= \pm \frac{\sqrt{3}}{2}
\end{align*}
\]

Thus, we need to solve two equations for $x$:

- $\sin x = \frac{\sqrt{3}}{2}$ and $\sin x = -\frac{\sqrt{3}}{2}$

In the first case, the inverse sine function as an operator yields $x = \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) \Rightarrow x = \frac{\pi}{3}$ as the primary solution, and by symmetry, $x = \frac{2\pi}{3}$ as the secondary solution.

In the second case, the inverse sine operator yields $x = \sin^{-1} \left( -\frac{\sqrt{3}}{2} \right) \Rightarrow x = \frac{4\pi}{3}$ as a primary solution and by symmetry, $x = \frac{5\pi}{3}$ as a solution. Thus, there are four solutions and the solution set is $\left\{ \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \right\}$.

**Example 5**: Solve for $x$ in the interval $0 \leq x < 2\pi$: $1 + \cos x = 2\sin^2 x$.

**Solution**: This equation involves both sine and cosine terms, as well as a squared term. In cases like this, it is usually best to get all trigonometric terms into a common form. Thus, we re-arrange this equation a bit, then replace the $\sin^2 x$ term using the identity $\sin^2 x = 1 - \cos^2 x$. This will result in a quadratic equation in terms of cosine only:
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\[ 1 + \cos x = 2\sin^2 x \]

\[ \Rightarrow -2\sin^2 x + 1 + \cos x = 0 \]

\[ \Rightarrow -2(1 - \cos^2 x) + \cos x + 1 = 0 \]

\[ \Rightarrow 2\cos^2 x + \cos x - 1 = 0 \]

The last equation is now a quadratic equation with \( \cos x \) as the unknown. Factor:

\[ 2\cos^2 x + \cos x - 1 = 0 \]

\[ \Rightarrow (2\cos x - 1)(\cos x + 1) = 0 \]

Each factor is then solved for zero:

\[ 2\cos x - 1 = 0 \quad \Rightarrow \quad \cos x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{3}, x = \frac{5\pi}{3} \]

and

\[ \cos x + 1 = 0 \quad \Rightarrow \quad \cos x = -1 \quad \Rightarrow \quad x = \pi \]

Therefore, the solution set is \( \left\{ \frac{\pi}{3}, \frac{5\pi}{3}, \pi \right\} \).

Other Forms

There is no limit to the number of possible equations involving trigonometric functions. Some may be easy to solve, some not so easy, and some impossible. You should always look for possible identities to simplify the equation, collecting terms, and avoiding the dividing out of a trigonometric function. The following example is one of many possible trigonometric equations:

Example 6: Solve for \( x \) in the interval \( 0 \leq x < 2\pi \): \( \sec^2 x = \csc x \tan x \)

Solution: Replacing each of these forms in terms of sine and cosine, we can greatly simplify this equation and write it in terms of a single trigonometric function:

\[ \sec^2 x = \csc x \tan x \]

\[ \Rightarrow \quad \frac{1}{\cos^2 x} = \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x} \]

\[ \Rightarrow \quad \frac{1}{\cos^2 x} = \frac{1}{\cos x} \]

Cross-multiplying and collecting the terms to one side yields \( \cos^2 x - \cos x = 0 \). We factor:

\[ \cos^2 x - \cos x = 0 \]

\[ \Rightarrow \quad \cos x(\cos x - 1) = 0 \]

Each factor is solved for its solutions:
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\[
\begin{align*}
\cos x = 0 & \implies x = \frac{\pi}{2}, \ x = \frac{3\pi}{2} \\
\cos x - 1 = 0 & \implies \cos x = 1 \implies x = 0
\end{align*}
\]

However, none of them are acceptable! Why is this? Note that the \( \sec x \) term is not defined when \( \cos x = 0 \), so the two solutions must be discarded, and the \( \csc x \) term is not defined when \( x = 0 \), so it too must be discarded. The solution set is empty: \( \emptyset \).

Example 7: Solve for \( x \) in the interval \( 0 \leq x < 2\pi \): \( \sin 2x + \sin 3x = 0 \)

Solution: These two sine terms are “unlike”; they cannot be combined conveniently by addition. Therefore, we use an appropriate sum-to-product identity (section 4.5) to assist:

\[
\sin 2x + \sin 3x = 0 \\
\implies 2\sin\left(\frac{5}{2} x\right)\cos\left(\frac{1}{2} x\right) = 0
\]

Therefore, each factor can be set to 0 and solved. We explore the solutions generated by the factor \( \sin\left(\frac{5}{2} x\right) = 0 \) first:

\[
\sin\left(\frac{5}{2} x\right) = 0 \implies \frac{5}{2} x = 0 + 2n\pi \quad & \quad \frac{5}{2} x = \pi + 2n\pi \implies \quad x = \frac{2}{5}(0 + 2n\pi) \quad & \quad x = \frac{2}{5}(\pi + 2n\pi)
\]

The solutions are generated by evaluating for \( n = 0, 1, 2, etc. \). By direct evaluation we get these solutions:

\[
\begin{align*}
\text{if } n = 0 & \implies x = 0 \\
\text{and } n = 1 & \implies x = \frac{4}{5}\pi \\
\text{and } n = 2 & \implies x = \frac{8}{5}\pi
\end{align*}
\]

The last solution \( \frac{10}{5}\pi \) is the same as \( 2\pi \) and can be ignored, and there is no need to evaluate for higher values of \( n \) since this will simply repeat our known solutions again.

The solutions generated by the factor \( \cos\left(\frac{1}{2} x\right) = 0 \), we get the following:

\[
\begin{align*}
\cos\left(\frac{1}{2} x\right) = 0 & \implies \frac{1}{2} x = \frac{\pi}{2} + 2n\pi \quad & \quad \frac{1}{2} x = \frac{3\pi}{2} + 2n\pi \implies \quad x = 2\left(\frac{\pi}{2} + 2n\pi\right) \quad & \quad x = 2\left(\frac{3\pi}{2} + 2n\pi\right)
\end{align*}
\]

The only meaningful solution that results from these two equations is \( x = \pi \). All others are outside the restriction \( 0 \leq x < 2\pi \). Therefore, the solution set to this equation is:

\[
\{0, \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}\}
\]

If \( f(x) = \sin 2x + \sin 3x \), then the solution set can be illustrated as the \( x \)-intercepts of this graph in the interval \( 0 \leq x < 2\pi \):
4.7 Law of Sines and Cosines

Law of Cosines
Suppose that a triangle has sides of lengths \( a, b, \) and \( c \) corresponding to the angles \( \alpha, \beta, \) and \( \gamma \) as shown in the figure.

Then
\[
\begin{align*}
a^2 &= b^2 + c^2 - 2bc \cos \alpha \\
b^2 &= c^2 + a^2 - 2ca \cos \beta \\
c^2 &= a^2 + b^2 - 2ab \cos \gamma
\end{align*}
\]

And Law of Sines given below
\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

Examples

1. A triangle has sides of length 8, and 6, and the angle between these sides is 30 degrees. Find the length of the third side.

Solution We use the formula (Use your calculator for simplification)
\[
a^2 = b^2 + c^2 - 2bc \cos \alpha \\
a^2 = 6^2 + 8^2 - 2(8)(6)\cos 30^\circ
\]
The third side has the length equals \( a \).

2. Use law of Sines to find the indicated side \( CB = x \), where angle \( CAB = 56 \) degrees and angle \( CBA = 65 \) degrees, \( AB = 27 \) cm.

Solution We have angle \( ACB = 180 - (56+65) = 59 \)
\[
\frac{\sin 59^\circ}{27} = \frac{\sin 56^\circ}{x}
\]
\[
x = \frac{\sin 56^\circ}{\sin 59^\circ} (27) = 26.11
\]