Chapter 1 Graph of Functions

1.1 Rectangular Coordinate System and Plotting points

The Coordinate Plane

![Figure 1.1](image1.jpg)

The axes divide the plane into four regions, called quadrants; those are numbered as in Figure 1.1. The point (1, 2) lies on quadrant I, the point (-4, -2) lies on quadrant III and so on.

Examples

1. Plot the points (3, 1), (-2, -1), (0, -3), (2, 0), (-5, 0) and (0, 2)

![Figure 1.2](image2.jpg)

1.2 Distance and Midpoint Formulas

The Distance formula: Consider two points P(a, b) and Q(c, d). Then we have the following measures \( \text{RUN} = PR = c - a \), and \( \text{RISE} = RQ = d - b \). Using Pythagorean
Theorem we have the distance between P and Q is \[ |PR| = \sqrt{(c-a)^2 + (d-b)^2} \]

Figure 1.3

The **distance** formula between two points \( A(x_1, y_1) \) and \( B(x_2, y_2) \) is given by

\[ |AB| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{\text{RUN}^2 + \text{RISE}^2} \]

2. Find the distance between the points (5, 7) and (-3, 8)

The distance \( d = \sqrt{(5+3)^2 + (7-8)^2} = \sqrt{65} \)

The **Midpoint** Formula: The midpoint of the line segment from \( A(x_1, y_1) \) and \( B(x_2, y_2) \) is given by

\[ \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \]

3. Find the midpoint of the line segment from (5, 4) to (3, 6)

**Solution:** The midpoint is (4, 5)

**1.3 Straight Lines and Slopes**

The well-known general form of a straight line is \( ax + by + c = 0 \) and the standard form of a straight line is \( y = mx + b \), also known as slope-intercept form. In the second form \( m \) is called the slope of the line.

Remember that if a straight line passes through \( A(x_1, y_1) \) and \( B(x_2, y_2) \) the slope is

\[ m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\text{RISE}}{\text{RUN}} \]

4. Find the slope of the straight line which passes through the points (7, 4) and (2, 3)

**Solution:** The slope is \( m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\text{RISE}}{\text{RUN}} = \frac{4-3}{7-2} = \frac{1}{5} \)
5. Find the slope of the straight line whose equation is given by $2x - 6y + 10 = 0$

**Solution**: we write the given equation $2x - 6y + 10 = 0$ as $y = \frac{1}{3} x + \frac{5}{3}$.

The slope is $m = \frac{1}{3}$.

**Properties of slopes**

The slope of non-vertical lines is a number $m$ that measures how steep are the lines. The following cases may arise:

- When $m = 0$, the straight line is horizontal
- When $m = 1$, the straight line makes 45 degree angle with the $x$ axis in the positive direction
- Larger the absolute value of the slope steeper the straight lines
- For a vertical line the slope is undefined
- The straight lines with positive slope are increasing and make acute angles with $x$ axis in the positive direction
- The straight lines with negative slopes are decreasing and make obtuse angles with $x$ axis in the positive direction

If $m_1$ and $m_2$ are the slopes of two given straight lines then the following cases may arise

- When $m_1 = m_2$, the straight lines are parallel
- When $m_1 \times m_2 = -1$, the straight lines are perpendicular

6. Plot the following straight lines and observe the slopes

   a) $y = x$
   b) $y = 2x$
   c) $y = .5x$
   d) $y = -x$
   e) $y = -2x$
   f) $y = -.5x$

**Solution**:  

![Graph of Functions](image)

Figure 1.4

7. Show that the straight lines $y = 10x + 3$ and $10y + x - 15 = 0$ are perpendicular to each other.

**Solution**: Observe that for $y = 10x + 3$ the slope is $m_1 = 10$ and on the other hand the slope of $10y + x - 15 = 0$ is $m_2 = -\frac{1}{10}$. We have $m_1 \times m_2 = -1$ and the straight lines are perpendicular.
8. Show that the straight lines \( y = 10x + 3 \) and \( 5y - 50x - 15 = 0 \) are parallel to each other.

**Solution:** The straight line \( y = 10x + 3 \) has slope \( m_1 = 10 \) and the straight line \( 5y - 50x - 15 = 0 \) has also the same slope \( m_2 = 10 \). The straight lines are parallel.

### 1.4 Basics of Functions

In everyday life we encounter situations in which one item is associated with another. For instance, the growth of a plant may depend on amount of fertilizer or the understanding of a problem may associate with hours of study.

**Relation and Function:**

A relation is a set of ordered pairs. A function is a relation in which no two ordered pairs have the same first coordinate. In the ordered pairs the set of all first coordinates constitutes the domain and the set of all second coordinates is the range of the relation.

**Formal definition of function**

A function is a rule that provides for every single value of \( x \) a corresponding unique value of \( y \). The domain of the function is the set of all permissible values of \( x \); on the other hand the range of the function is the set of all possible values of \( y \).

**Vertical line test to determine a function**

A graph defines a function if any vertical line intersects the graph at no more than one point.

a) Following are the graphs of functions

![Graphs of Functions](Fig 1.5, Fig 1.6, Fig 1.7)

b) Following are not the graphs of functions

![Graphs that are not functions](Fig 1.8, Fig 1.9, Fig 1.10)

### 1.5 Graph of Functions

9. Plot the function \( y = f(x) = x \)
1 Graph of Functions

**Solution:** This function is called an identity function. For every single value of $x$ we have a unique value of $y$ equals $x$. Thus following are the points on the graph of the function.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = f(x)$</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1.1

![Graph of identity function](image)

Fig 1.11

10. Plot the function $x = 2$

**Solution:** It is a vertical straight-line at 2 units to the right from the origin. You try to make a plot.

11. Plot the function $y = 3x + 5$

**Solution:** Let us consider the following points

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = f(x)$</td>
<td>-1</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 1.2

![Graph of linear function](image)

Fig 1.12

12. Plot the quadratic function $y = x^2 - 1$
1.6 Transformation of Functions

In this section you will observe when the rule of a function is algebraically changed in certain ways to get a new function, then the graph of the new function can be obtained from the graph of the original function by simple geometric transformations. Below we discuss horizontal and vertical shifting.

**Vertical shift**
Consider the original function \( y = f(x) = x^2 \) and its graph then compare with two other functions \( y = g(x) = x^2 - 1 = f(x) - 1 \) and \( y = h(x) = x^2 + 1 = f(x) + 1 \). Look at the graphs below. And notice that the graph of \( y = g(x) \) is found from the graph of \( y = f(x) \) by shifting it vertically 1 unit downward. On the other hand the graph of \( y = h(x) = x^2 + 1 = f(x) + 1 \) is found from the graph of \( y = f(x) \) by shifting it vertically 1 unit upward.
**Rule on vertical shift:** The graph of \( y = g(x) = f(x) + k \) is found by shifting the graph of \( y = f(x) \) vertically by \( k \) units. If \( k \) is positive then the shift is upward and if it is negative the shift is downward.

**Horizontal shift**

Consider the original function \( y = f(x) = x^2 \) and its graph then compare with two other functions \( y = g(x) = (x-1)^2 = f(x-1) \) and \( y = h(x) = (x+1)^2 = f(x+1) \). Look at the graphs below. And notice that the graph of \( y = g(x) \) is found from the graph of \( y = f(x) \) by shifting it horizontally 1 unit to the right. On the other hand the graph of \( y = h(x) \) is found from the graph of \( y = f(x) \) by shifting it horizontally 1 unit to the left.

**Rule on horizontal shift:** The graph of \( y = g(x) = f(x+h) \) is found by shifting the graph of \( y = f(x) \) horizontally by \( h \) units. If \( h \) is positive then the shift is to the left and if it is negative the shift is on right.

**Reflection**

The graph of \( y = g(x) = -f(x) \) is found by reflecting the graph of \( y = f(x) \) along \( x \)-axis.

![Graph of Functions](image)

![Rule on vertical shift](image)

![Horizontal shift](image)

![Rule on horizontal shift](image)

![Reflection](image)
**Stretch** (Elongate or expand) and **Shrink** (Compress or contract)

The following diagram is useful to remember the stretch and shrink of a graph by a known factor \( c \), with the following values. We consider the cases \( y = f(cx) \) for horizontal stretch (\( H_{ST} \)) or horizontal shrink (\( H_{SH} \)) and \( y = cf(x) \) for vertical stretch (\( V_{ST} \)) or vertical shrink (\( V_{SH} \)).

\[
\begin{array}{c|cc}
|c| & H_{ST} & H_{SH} \\
\hline
|c| < 1 & 1 & |c| > 1 \\
V_{SH} & V_{ST}
\end{array}
\]

**Rule on vertical stretch and shrink**: The graph of \( y = g(x) = cf(x) \) is found by
- Shrinking vertically the graph of \( y = f(x) \) by a factor of \( c \) when \( |c| < 1 \)
- Stretching vertically the graph of \( y = f(x) \) by a factor of \( c \) when \( |c| > 1 \)

Note: Remember that for \( y = g(x) = cf(x) \), when \( c \) is negative we have horizontal reflection or reflection along x-axis.

**Rule on horizontal stretch and shrink**: The graph of \( y = g(x) = f(cx) \) is found by
- Stretching horizontally the graph of \( y = f(x) \) by a factor of \( 1/c \) when \( |c| < 1 \)
- Shrinking horizontally the graph of \( y = f(x) \) by a factor of \( 1/c \) when \( |c| > 1 \)

Note: Remember that for some cases of \( y = g(x) = f(cx) \), when \( c \) is negative we have vertical reflection or reflection along y-axis.

a) Compare the graphs of \( y = f(x) = x^2 \), \( y = g(x) = (2x)^2 = f(2x) \) and \( y = g(x) = (0.5x)^2 = f(0.5x) \). Observe that the graph of \( y = f(x) = x^2 \) is **horizontally shrunk** by a factor of \( c = 1/2 \) for the \( x \) coordinates (divide \( x \) coordinates by 2 or multiply by 1/2) to get the graph of \( y = g(x) = (2x)^2 = f(2x) \), while the graph of \( y = f(x) = x^2 \) is **horizontally stretched** by a factor of \( c = 0.5 \) for the \( x \) coordinates (divide \( x \) coordinates by 0.5 or multiply by \( 1/0.5 = 2 \)) to get the graph of \( y = g(x) = (0.5x)^2 = f(0.5x) \).
b) Compare the graphs of \( y = f(x) = x^2 \), \( y = g(x) = 2x^2 = 2f(x) \) and \( y = h(x) = 0.5x^2 = 0.5f(x) \). Observe that the graph of \( y = f(x) = x^2 \) is \textit{vertically stretched} by a factor of \( c = 2 \) for the \( y \) coordinates (multiply \( y \) coordinate by 2) to get the graph of \( y = g(x) = (2x)^2 = f(2x) \). On the other hand the graph of \( y = f(x) = x^2 \) is \textit{vertically shrunk} by a factor of \( c = 0.5 \) for the \( y \) coordinates (multiply \( y \) coordinate by 0.5) to get the graph of \( y = g(x) = (0.5x)^2 = f(0.5x) \).

\[
\begin{align*}
\text{Fig 1.18} \\
y = g(x) = (2x)^2 \\
y = f(x) = x^2 \\
y = g(x) = (0.5x)^2
\end{align*}
\]

**Determining a function obtained from a series of transformations**

13. Discuss the transformation of \( g(x) = \frac{4}{x-1} + 1 \) compared to \( f(x) = \frac{1}{x} \).

**Solution:** We follow the following steps: Step 1. Get \( 4f(x) = \frac{4}{x} \), \( c = 4 \), the graph is vertically stretched by a factor of 4 (multiply \( y \) coordinate by 4)

Step 2. Get \( 4f(x-1) = \frac{4}{x-1} \), from step 1, the graph moves 1 unit horizontally to the right

Step 3. Get \( 4f(x-1)+1 = \frac{4}{x-1} + 1 \), from Step 2, the graph moves 1 units upward.

\[
\begin{align*}
\text{Fig 1.19} \\
\text{We get second graph (F2) multiplying } y \text{ coordinate by 4, then move the graph horizontally by positive 1 unit to get third graph (F3) and finally 1 unit upward to get the fourth graph (F4), which is the final position.}
\end{align*}
\]
1.7 Combination of Functions, Composite Functions

Combinations of Functions

In this section we will discuss about addition (sum), subtraction (difference), multiplication (product) and division (quotient) of functions. We also further discuss an important operation called composition of functions. Suppose \( f(x), g(x) \) are two functions, then their sum is the function say \( h(x) \) defined as

\[
h(x) = f(x) + g(x) = (f + g)(x)
\]

and the difference \( k(x) = f(x) - g(x) = (f - g)(x) \).

In the same way the other defined functions are \( m(x) = f(x) g(x) = (fg)(x) \) is a function from multiplication and \( n(x) = f(x) / g(x) = (f / g)(x), g(x) \neq 0 \) is a function from division.

The domain of all these combinations of functions is the common domain to both the functions. If \( D_f \) is the domain of \( f(x) \), and \( D_g \) is the domain of \( g(x) \), then the common domain is \( D_f \cap D_g \). Note that for \( f(x) / g(x), g(x) \neq 0 \), the domain is \( D_f \cap D_g \) when \( g(x) \neq 0 \).

14. For the functions \( f(x) = 2x^4 + x^2 \) and \( g(x) = \sqrt{x} \) determine the following functions and their domain

\[
a) \ f(x) + g(x) \quad b) \ g(x) - f(x) \quad c) \ g(x) f(x) \quad d) \ \frac{f(x)}{g(x)}, g(x) \neq 0
\]

**Solution:**

a) \( f(x) + g(x) = 2x^4 + x^2 + \sqrt{x} \), the common domain is \([0, \infty)\)

b) \( g(x) - f(x) = \sqrt{x} - 2x^4 - x^2 \), the common domain is \([0, \infty)\)

c) \( g(x) f(x) = \sqrt{x}(2x^4 + x^2) \), the common domain is \([0, \infty)\)

d) \( \frac{f(x)}{g(x)} = \frac{2x^4 + x^2}{\sqrt{x}}, g(x) \neq 0 \), the common domain is \((0, \infty)\)

Composite function or composition of functions

For two given functions \( f(x), g(x) \), the compositions are defined as

\( f(x) \circ g(x) = f \circ g = f(g(x)) \) and \( g(x) \circ f(x) = g \circ f = g(f(x)) \)

The domain of \( f \circ g \) is defined as i) \( x \) is in \( g(x) \) and ii) \( g(x) \) is in \( f(x) \)
15. For the functions \( f(x) = 2x^3 + x^2 \) and \( g(x) = \sqrt{x} \) determine the following functions and their domain

a) \( f(x) \circ g(x) \)  

b) \( g(x) \circ f(x) \)

**Solution:**

a) \( f(x) \circ g(x) = f(g(x)) = f(\sqrt{x}) = 2x^2 + x \) has domain all \( x \geq 0 \)

b) \( g(x) \circ f(x) = g(f(x)) = g(2x^3 + x^2) = 2x^4 + x^2 \) has domain all \( x \) such that \( 2x^4 + x^2 \geq 0 \) which is true for all real values of \( x \). Thus the domain is the entire real line.

16. For the functions \( f(x) = \frac{1}{x - 1} \) and \( g(x) = \frac{1}{x + 1} \) determine the following functions and their domain

a) \( f(x) \circ g(x) \)  

b) \( g(x) \circ f(x) \)

**Solution:**

a) \( f(x) \circ g(x) = f(g(x)) = f\left(\frac{1}{x + 1}\right) = \frac{1}{\frac{1}{x + 1} - 1} = \frac{x + 1}{x} \)

The domain is \( x \neq -1 \) which is domain for \( g(x) = \frac{1}{x + 1} \) and also for \( g(x) \) should be defined on \( f(x) \) when \( g(x) = \frac{1}{x + 1} \neq 1 \Leftrightarrow x \neq 0 \). Thus the domain is all real values of \( x \neq -1, 0 \)

b) \( g(x) \circ f(x) = g(f(x)) = g\left(\frac{1}{x - 1}\right) = \frac{1}{\frac{1}{x - 1} + 1} = \frac{x - 1}{x} \)

The domain is \( x \neq 1 \) which is domain for \( f(x) = \frac{1}{x - 1} \) and also for \( f(x) \) should be defined on \( g(x) \) when \( f(x) = \frac{1}{x - 1} \neq -1 \Leftrightarrow x \neq 0 \). Thus the domain is all real values of \( x \neq 1, 0 \)

1.8 One-to-one and Inverse Functions

**One-to-one function**  
If the inverse of any function is also a function then the function must be one-to-one. For a one-to-one function \( f(x) \), each \( x \) in the domain has only one image in the range. No \( y \) in the range is the image of more than one \( x \) in the domain. Thus if a function \( f(x) \) has an inverse it is one-to-one. And when the function \( f(x) \) is one-to-one then for any element \( x_1 \) and \( x_2 \) in the domain of \( f(x) \) with \( x_1 \neq x_2 \) we will have \( f(x_1) \neq f(x_2) \). A one-to-one function can be determined by horizontal line test.
1 Graph of Functions

**Horizontal line test** If every horizontal line intersects the graph of $y = f(x)$ in at most one point, then the function is one-to-one.

![Graphs of Functions](image)

One-to-one function

One-to-one function

Not one-to-one

17. Graph the following functions and determine if it is a one-to-one function.

a) $f(x) = x^3 - 2x^2 + 9x - 7$

b) $f(x) = x^3 - 9x - 7$

c) $f(x) = -x^3 - 7$

**Solution:**

![Graphs of Functions](image)

One-to-one function

Not one-to-one

one-to-one

In Fig 1.25 the horizontal line $y = 7$ seems to have infinitely many $x$ values very close to zero, still it considered to be one-to-one.

**Inverse Functions** Two one-to-one functions $f(x)$ and $g(x)$ are said to be inverses to each other if $f(x) \circ g(x) = g(x) \circ f(x) = x$ and we write $f(x) = g^{-1}(x)$. Note that $f(f^{-1}(x)) = x$. If $f(x)$ has domain $D$ and range $R$ then its inverse $g(x)$ has domain $R$ and range $D$.

If $g(x)$ is the inverse of $f(x)$, then the graph of $g(x)$ is the reflection of $f(x)$ in the line $y = x$, known as the line of symmetry.

18. We consider a one-to-one function $f(x) = 3 + 2x$ and few points. Also suppose that $g(x)$ is the inverse of $f(x)$. We have following results

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>-1</th>
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</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>3</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
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<th>5</th>
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</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

![Graphs of Functions](image)

Fig 1.26
Determining the inverse of a one-to-one function $f(x)$

- Set $y = f(x)$
- Interchange the variables $x$ and $y$ so that the domain of inverse of $f(x)$ is represented by numbers on the $x$-axis.
- Solve for $y$ and replace $y$ by $f^{-1}(x)$

19. Find the inverse of $f(x) = 3 + 2x$

**Solution:** $f(x) = 3 + 2x$

- $y = 3 + 2x$
- $x = 3 + 2y$
- $y = \frac{x - 3}{2}$
- $f^{-1}(x) = \frac{x - 3}{2}$

20. Find the inverse of $f(x) = \frac{3 + 2x}{x - 1}$

**Solution:** We write $y = \frac{3 + 2x}{x - 1}$, domain is all $x \neq 1$ and range is all $y \neq 2$

Replace $x$ by $y$ and $y$ by $x$ to get $x = \frac{3 + 2y}{y - 1}$

Solve for $y$:

\[
x = \frac{3 + 2y}{y - 1}
\]
\[
3 + 2y = xy - x
\]
\[
y(x - 2) = 3 + x
\]
\[
y = \frac{3 + x}{x - 2} = f^{-1}(x)
\]

has domain with all $x \neq 2$ and range with all $y \neq 1$.

21. Find the inverse of $f(x) = \sqrt{2 - x}$

**Solution:** We write $y = \sqrt{2 - x}$, domain is all $x > 0$ and range is $y \leq 2$

Replace $x$ by $y$ and $y$ by $x$ to get $x = \sqrt{2 - y}$

Solve for $y$:

\[
x = \sqrt{2 - y}
\]
\[
x^2 = 2 - y
\]
\[
y = 2 - x^2 = f^{-1}(x)
\]