Chapter 10 Markov Chains and Transition Matrices

A Markov Chain is a sequence of experiments, each of which results in one of a finite number of states that we label 1, 2, 3, .. m. If $p_{ij}$ is the probability of moving from state i to state j, then the transition matrix $P = [p_{ij}]$ of a Markov chain is the $m \times m$ matrix

$$
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1m} \\
p_{21} & p_{22} & \cdots & p_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
p_{(m-1)1} & p_{(m-1)2} & \cdots & p_{(m-1)m} \\
p_{m1} & p_{m2} & \cdots & p_{mm}
\end{bmatrix}
$$

Note that the sum of entries of each row is 1, as those represent probabilities of any state. $v^{(0)} = [p_{11} \ p_{12} \ p_{13} \ \cdots \ p_{1m}]$ is an initial probability distribution.

Probability distribution after k stages has the following relation

$$v^{(k)} = v^{(k-1)}P = v^{(0)}P^k$$

Example 1. Consider the transition matrix given below and answer following questions:

$$P = \begin{bmatrix}
1/3 & 2/3 \\
1/4 & 3/4
\end{bmatrix}$$

a) What does the entry 3/4 in this matrix represent?
b) Assuming that the system is initially in State 1, find the probability distribution one observation later. What is it two observations later?
c) Assuming that the system is initially in State 2, find the probability distribution one observation later. What is it two observations later?

Solution: a) The transition probability from state 2 to state 2 is 3/4. In other words, it the probability that an object in state 2 will move to state 2.

b) $v^{(0)} = [1 \ 0]$, $v^{(1)} = v^{(0)}P = [1/3 \ 2/3]$ one observation later,

$$v^{(2)} = [5/18 \ 13/18]$$ two observation later
c) $v^{(0)} = [0 \ 1], \ v^{(1)} = v^{(0)}P = [1/4 \ 3/4]$ one observation later,  
$v^{(2)} = [13/48 \ 35/48]$ two observations later

Example 2. $P = \begin{pmatrix} .7 & .2 & .1 \\ .6 & .2 & .2 \\ .4 & .1 & .5 \end{pmatrix}$, $v^{(0)} = [.3 \ .3 \ .4]$. Determine $v^{(1)}, v^{(2)}$

Homework #3 (Example 3) due on Wednesday Jan 9, 2008

Example 3. The city of Oak Lawn is experiencing a movement of its population to suburbs. At present 70% of the total population lives in the city and 30% lives in the suburbs. But each year 7% of the city people move to suburbs while only 1% of the suburbs people move back to the city. Assuming that the total population (city and suburbs) remains constant. Find what is the ratio of city people to suburbs people after 5 years. 
(Answer: Approximately 1 to 1)

Example 4. Consider a maze consisting of four rooms as shown in the figure. The experiment consists of releasing a mouse in a particular room and observing its behavior. Consider the transition matrix

\[
P = \begin{pmatrix}
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\
\frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4}
\end{pmatrix}
\]

In a certain case if the initial probability distribution is $v^{(0)} = [1/6 \ 2/3 \ 0 \ 1/6]$. Find probability distribution after 5 observations.

Example 5. Consider a maze in 9 rooms. The system consists of releasing a mouse in any room. We assume that the following learning patterns exist. If the mouse is in the rooms 1, 2, 3, 4, 5, it remains in that room or it moves to any room that the maze permits, each having the same probability; if it is in room 8, it moves directly to room 9; if it is in rooms 6, 7, or 9, it remains in that room. The mouse can move from 1 to 4 only; from 4 to 1 or to 7; from 2 to 5 only; from 5 to 2 or to 8; 3 to 6 only; 6 to 3 or to 9; and 8 to 9 only; 9 to 6 or to 8. Draw the maze and answer the following questions:

a) Explain why the above experiment is a Markov chain.
b) Construct the transition matrix $P$.
c) Find the transition probability after the third state, given the initial probability matrix $v^{(0)} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1]$. 

**Regular Markov Chains:** A Markov chain is said to be regular if, for some power of its transition matrix, all of the entries are positive. Remember that here zero is not considered as positive. For this section, zero is zero, neither positive nor negative.

**Example 6.** \( P = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \) is regular since \( P^2 = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} \)

**Example 7.** \( P = \begin{bmatrix} 1 & 0 \\ 3/4 & 1/4 \end{bmatrix} \) is not regular. Verify.

**Example 8.** \( P = \begin{bmatrix} 3/4 & 1/4 \\ 7/20 & 13/20 \end{bmatrix} \). Find \( P^{12} \). Consider \( t = [0.5833 \ 0.4167] \)

And show that \( t = tP \). In this case we say that, the row vector \( t \) is fixed by \( P \).

**Example 9.** Find the fixed probability vector \( t \) of the Markov Chain whose transition matrix is \( P = \begin{bmatrix} 0.93 & 0.07 \\ 0.01 & 0.99 \end{bmatrix} \)

**Example 10.** Is the transition matrix \( P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \) regular?

**Absorbing Markov Chain:** In a Markov Chain if \( p_{ij} \) denotes the probability of going from state \( E_i \) to state \( E_j \), then \( E_i \) is called absorbing state if \( p_{ij} = 1 \). A Markov Chain is called absorbing Markov Chain iff it contains at least one absorbing state and it is possible to go from any non-absorbing state to an absorbing state in one or more stages.

**Example 11.** \( P = \begin{bmatrix} 1 & 0 \\ 3/4 & 1/4 \end{bmatrix} \) is absorbing as first row is absorbing state.

**Subdividing** transition matrices to absorbing Markov Chain: Any matrix subdivided as the given form is such a matrix,

\[ P = \begin{bmatrix} I & O \\ S & Q \end{bmatrix} \]

where \( I \) is the identity matrix, \( O \) is the zero matrix.
Example 12. Subdivide the matrix as absorbing Markov Chain: 
\[
P = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}
\]

Two person games: Any conflict or competition between two people is called a two-person game.

Example 13. From tossing a coin, player I picks heads or tails and player II attempts to guess the choice. Player I will pay player II $3 if both choose heads; player I will pay player II $2 if both choose tails. If player II guesses incorrectly, he will pay player I $5. Write the payoff matrix.

Strictly determined game: A game defined by a matrix is said to be strictly determined iff there is an entry of the matrix that is the smallest element in its row and is also the largest in its column. This entry is then called the value of the game, which is saddle point.

Example 14. Determine whether the game defined by the matrix below is strictly determined. If it is find the value of the game.

\[
a) \quad P = \begin{bmatrix} 3 & 0 & -2 & -1 \\ 2 & -3 & 0 & -1 \\ 4 & 2 & 1 & 0 \end{bmatrix} \\
b) \quad P = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 6 & 0 \\ 1 & 3 & 7 \end{bmatrix} \\
c) \quad P = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}
\]

Mixed strategy: Whenever strategies are mixed for players, it is called mixed strategy game.

Expected Payoff: The expected payoff \( E \) of player I in a two person zero sum game, defined by the matrix \( A \), in which the row vector \( P \) and column vector \( Q \) define the respective strategy probabilities of player I and player II is 
\[
E = PAQ
\]

Example 15. Given \( A = \begin{bmatrix} 3 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix} \), \( P = [1/3 \; 1/3 \; 1/3] \), \( Q = [1/3 \; 2/3] \), Find \( E \)

Solution: 
\[
PAQ = \begin{bmatrix} 3 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = [2/9]
\]

is biased in favor of player I and has an expected payoff of 2/9.
Example 16. Given that expected payoff \( E \) corresponding to the optimal strategies is given by

\[
E = PAQ = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} ; \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\]

Question: For the game matrix

\[
A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}
\]

find the expected payoff.

Transition diagram: The graphical representation of a transition matrix is called the transition diagram.

Example 17. An insurance company classifies drivers as low-risk if they are accident-free for 1 year. Past records indicate that 98% of the drivers in the low-risk category (L) one year will remain in that category the next year, and 78% of the drivers who are not in the low-risk category (L’) one year will be in the low-risk category the next year.

a) Draw a transition diagram
b) Find the transition matrix \( P \)
c) If 90% of the drivers in the community are in the low-risk category this year, what is the probability that a driver chosen at random from the community will be in the low-risk category next year? Year after next?

\[
P = \begin{bmatrix} .98 & .02 \\ .78 & .22 \end{bmatrix}
\]

a) Transition Matrix

b) Transition diagram

\[
\begin{array}{ccc}
L & L' \\
\cdot98 & .02 \\
\cdot78 & .22
\end{array}
\]

Therefore 96% of the drivers will be in the low-risk category next year and 97.2% of the drivers will be in the low-risk category year after next year.

Example 18. Part-time students admitted to an MBA program in a university are considered to be first-year students until they complete 15 credits successfully. Then they are classified as second-year students and may begin to take more advanced courses and to work on the thesis required for graduation. Past record indicate that at the end of each year 10% of the first-year students (F) drop out of the program (D) and 30% become second-year students (S). Also, 10% of the second-year students drop out of the program and graduate (G) 40% each year. Students that graduate or drop out never return to the program.

a) Draw a transition diagram.

b) Find the transition matrix \( P \)
c) What is the probability that a first-year student graduates within 4 years? Drop out within 4 years?

Example 19. Find transition matrix from the given transition diagram (complete the transition diagram before finding transition matrix):