Kinetic Theories of Vehicular Traffic

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Ingredients of Kinetic Models

• Mechanical model
  – Microscopic model giving response of following vehicle to specified speeds of itself and its leader, and spatial headway
  – Such models of interest in themselves

• Correlation model
  – Approx. of follower-leader pairwise distribution in terms of single-vehicle distribution
Ingredients of P-H Kinetic Model

• *If follower faster:*
  – **Mechanical model** is speed persistence up to just-in-time braking to avoid collision
  – **Correlation model** is “vehicular chaos” (statistical independence of follower and leader)

• *Otherwise:*
  – **Both models** represented by phenomenological relaxation term
  – Vehicular chaos definitely unsuitable
The Prigogine-Herman Equation

\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = c(\nu - v)(1 - P)f - \frac{f - f_0}{T} \]

streaming = slowing down + relaxation

\[ f = f(x, v, t) \]

\[ f_0 = \text{desired speed distribution} \]

\[ c, cv = q \text{ zeroth and first moments of } f \text{ wrt } v \]

\[ P \text{ and } T \text{ known (perhaps in terms of } c) \]
Existence-Uniqueness for P-H

• Local E-U is relatively straightforward
  – semigroup theory (Barone and Bellini-Morante, 1978)
  – contractive mapping, à la Grad

• Global theory seems to have many of the well-known complications of the Boltzmann equation
  – 1-d simplifications?
Criticisms of P-H

• The P-H equation has been widely criticized, some of it constructively
  – Munjal and Pahl (1969) [validity of slowing-down term in presence of platoons; phenomenological nature of relaxation term]
  – Paveri-Fontana (1975) [desired speed should be additional independent variable]
  – See *Transportation Research, 9* (1975), 225 for P-F extension of P-H model.
Three Post-95 Threads

• Klar, Wegener et al. developed models that treat speeding up more fundamentally
  – Enskog effects

• Helbing et al. in Germany and Hoogendoorn and Bovy in Holland employed the P-F model
  – e.g., *Transportation Research B*, **35** (2001), 317.

• A. Sopasakis and PN studied P-H model *per se*
  – * Principally for purpose of establishing baseline*
Consistency Questions

• Does validity of \( \int_{v}^{v_{\text{max}}} f(x, v', t)dv' \leq \int_{v}^{v_{\text{max}}} f_{0}(x, v', t)dv' \) for \( t = 0 \) and all \( v \in [0, v_{\text{max}}] \) imply the same equality for all \( t > 0 \)? (J. J. Dorning, 1999)

• Similarly, does \( \int_{v}^{v_{\text{max}}} v' f(x, v', t)dv' \leq \int_{v}^{v_{\text{max}}} v' f_{0}(x, v', t)dv' \) for \( t = 0 \) and all \( v \) imply the same inequality for all \( t > 0 \)? (Relaxation term = speeding up?)

• Similar consistency question for slowing-down term.
Eq. Solns. Of P-H

- Expect one-parameter (c) family
- \( \rightarrow \) flow = \( q = Q(c) = \) traffic stream model (fundamental diagram)
- TSM + continuity \( \rightarrow \) LW model
- P&H show this expectation holds, \textit{if} desired speed distribution > 0 for arbitrarily small positive speeds
- Otherwise
There exists $c_{\text{crit}} > 0$ s. t.:
- if $0 < c \leq c_{\text{crit}}$ the eq. solns. are a one-parameter family $f_{eq}(v;c)$;
- but if $c > c_{\text{crit}}$ the eq. solns. are the two-parameter family

$$f_{eq}(v; c, \zeta) = \frac{1}{cT (1 - P)(v - \zeta)} f_0 + \alpha(\zeta) c \delta(v - \zeta)$$

where $0 \leq \zeta < v_{\text{min}}$, with $\zeta$ the speed of the embedded collective flow.
Eq. Solns. Of P-H (3)

• Bimodal distribution, for $c > c_{crit}$
  – upper mode distributed across $[v_{min}, v_{max}]$, more weighted toward lower end as $\zeta \rightarrow v_{min}$
  – lower mode = $\delta$-function concentrated at speed $\zeta < v_{min}$

• This bimodality supported by data from Phillips:
Fig. 15. The Equilibrium Speed Distribution, $K = 96$ Cars/mi (USH1000)
• Note that now

\[
q = \begin{cases} 
\int v f_{eq}(v; c) dv = Q(c) \text{ for } c \leq c_{\text{crit}}, \text{ but} \\
\int v f_{eq}(v; c, \zeta) dv = Q(c, \zeta) \text{ for } c > c_{\text{crit}}.
\end{cases}
\]
The Corresponding 3-D TSM
2-D Projection of the 3-D TSM
Observational Data: Another Validation Point?

• Agrees qualitatively with oft-observed wide scatter in flow/density data, at densities $> \sim 35 \text{ vpm} (\sim 20 \text{ vpkm})$
  – e.g., the following observational data from Drake, Schofer and May (1967)

• For more details see PN and A. Sopasakis, *Transportation Research B*, 31 (1998), 589.
Data from Drake, Schofer and May

Figure 27. Volume-density relationship—Edie hypothesis.
From Kinetic to Macroscopic Models

• “Continuum” is preferred to “macroscopic”
    (http://www.tfhrc.gov/its/tft/tft.htm)
Sidebar: Views of Models

• Real traffic $\approx$ mathematical model
  – Validation (mathematicians)

• Mathematical model $\approx$ computational model
  – Verification (mathematicians)

• Real traffic $\approx$ computational model
  – Calibration (transportation specialists)
Chapman-Enskog Expansion

• C-E, applied to P-H below \( c_{\text{crit}} \), gives:
  – At zero order a L-W approximation
  – At first order a diffusively corrected L-W approximation, \( q = Q_0(c) - D(c)c_x \).
  – At second order a similar result, but with flow a complicated nonlinear function of spatial derivatives up to order 2 (A. Sopasakis)

• Diffusively corrected L-W also follows from macroscopic arguments (anticipation)
  – Unlike so-called higher-order models, not subject to flow reversal (PN, tentatively accepted)
Benchmarks for Kinetic Models

• **BM 1:** Eq. soln. should be *bimodal* at high densities

• **BM 2:** Corresponding TSM should display observed *scatter* at high densities

• **BM 3:** First-order *C-E* solution should be arguably reasonable improvement on L-W
# Kinetic Models vs. Benchmarks

<table>
<thead>
<tr>
<th>Model</th>
<th>bimodal?</th>
<th>unstable?</th>
<th>C-E?</th>
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<tbody>
<tr>
<td>Prigogine-Herman</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>Paveri-Fontana</td>
<td>NE</td>
<td>NE</td>
<td>NE</td>
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<tr>
<td>Phillips</td>
<td>no?</td>
<td>no?</td>
<td>no</td>
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<tr>
<td>P-H (Nelson-Sopasakis)</td>
<td>yes</td>
<td>yes</td>
<td>yes?</td>
</tr>
<tr>
<td>Illner-Klar-Materne(VFP)</td>
<td>yes?</td>
<td>yes</td>
<td>no?</td>
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Conclusions

• Among presently proposed kinetic models, P-H best meets the three benchmarks
  – (w/ appropriate desired speed distribution)
  – but the work of Klar et al. is close

• Macroscopic models of traffic flow face heavy burden of mistrust
  – originally from misuse of computational approximations
  – reinforced by feasibility, intuitive nature and ready graphical interpretation of microscopic models