Noise and \(O(1)\) amplitude effects on heteroclinic cycles

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The dynamics of structurally stable heteroclinic cycles connecting fixed points with one-dimensional unstable manifolds under the influence of noise is analyzed. Fokker-Planck equations for the evolution of the probability distribution of trajectories near heteroclinic cycles are solved. The influence of the magnitude of the stable and unstable eigenvalues at the fixed points and of the amplitude of the added noise on the location and shape of the probability distribution is determined. As a consequence, the jumping of solution trajectories in and out of invariant subspaces of the deterministic system can be explained. © 1999 American Institute of Physics.

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I. INTRODUCTION

Recently, it has been recognized that some dynamical systems are very susceptible to noise.\(^1\)\(^-\)\(^3\) In these cases, the addition of very small amplitude noise to the vector field leads to \(O(1)\) amplitude or time scale effects. As a result, one finds qualitatively different behavior in noisy systems than what would have been expected from the purely deterministic system. We will discuss here similar \(O(1)\) effects of noise near heteroclinic cycles which result from a subtle interplay between the parameters of the dynamical system and its susceptibility to noise.

The specific example that we are concerned with is the dynamics near heteroclinic cycles which are structurally stable. These heteroclinic cycles are characterized by the fact that due to an equivariance of the vector field under some symmetry group, there exist invariant subspaces in which saddle-saddle connections are structurally stable. The prototypical example is shown in Fig. 1, where there are three saddle points on the coordinate axes, and where all coordinate planes are invariant subspaces. Hence any coordinate plane has a saddle-sink connection and the union of all three of these connections comprises a heteroclinic cycle. Assuming that the heteroclinic cycle is asymptotically stable, solutions will approach the cycle and spend longer and longer times near the saddle points. Deterministically, in the limit \(t \to \infty\), the time a trajectory spends in the neighborhood of any saddle will go to infinity. However, as has been pointed out by Busse and Heikes\(^4\) and analyzed by Stone and Holmes,\(^5\) this is not what is expected in simulations or in real physical systems that are modeled by such a dynamical system. In these cases, noise near the saddle fixed points will prevent the trajectories from staying an indefinite amount of time in a neighborhood of such a saddle. This leads to what has been called a “statistical limit cycle” characterized by an intermittent time signal.

Figures 2 and 3 show two typical time simulations with noise. We observe that the system has a mean residence time near any of the saddle fixed points characterized by one of the variables being close to one and the other two variables being close to zero. In addition, we see that variables change sign occasionally. This is also a result of the added noise since every coordinate plane in the system is invariant and hence the deterministic trajectory never changes sign in any coordinate. However, there is a curious difference in the two simulations: while in Fig. 2 all coordinate planes are crossed, only one coordinate changes sign in Fig. 3.

The analysis in Ref. 2 examines the effect of very small amplitude noise on the period of a structurally stable cycle. It is assumed there that the noise has negligible effect on the amplitude of the cycle. In the systems studied here we consider when perturbing a structurally stable cycle with very small amplitude noise has \(O(1)\) effects on the amplitude of the cycle. Depending on the spectrum of eigenvalues of the saddle points, we will see that tiny noise can be amplified by the dynamics and appear in the period, or the period and the amplitude of the cycle.

This paper will develop a theory explaining the behavior of noisy heteroclinic cycles connecting fixed points with one-dimensional unstable manifolds. We solve the Fokker-Planck equations for the evolution of the probability distribution of trajectories near the heteroclinic cycle. The resulting probability distributions will depend on the noise level

and the eigenvalues of the saddles involved in the heteroclinic cycle. We find that the probability distribution just past a saddle fixed point is centered on the heteroclinic trajectory if the absolute value of the stable eigenvalue \( \lambda_s \) of the previous saddle is larger than the least unstable eigenvalue \( \lambda_u \). The probability distribution becomes skewed and centered away from the coordinate plane if \( \lambda_s < \lambda_u \). We are then able to predict the spread of the noisy trajectories about the heteroclinic orbits that results in hopping out of and into invariant subspaces.

In Sec. II we introduce our model system, we review the relevant stochastic methods in Sec. III, and apply them to our model system in Sec. IV. We conclude with some proposed extensions of our work.

II. THE STATISTICAL LIMIT CYCLE

In 1980 Busse and Heikes\(^4\) introduced a set of three coupled real ordinary differential equations (ODEs) to describe the behavior of the Küppers-Lortz instability of rotating convection rolls. The set of equations

\[
\begin{align*}
\dot{x}_1 &= x_1(q_0 + q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2), \\
\dot{x}_2 &= x_2(q_0 + q_3 x_2^2 + q_1 x_1^2 + q_2 x_3^2), \\
\dot{x}_3 &= x_3(q_0 + q_2 x_3^2 + q_1 x_1^2 + q_3 x_2^2),
\end{align*}
\]

possesses reflection and permutation symmetry of the three coordinate planes/axes. By using these symmetries it can be shown that a structurally stable heteroclinic cycle exists connecting the three fixed points located on the coordinate axes\(^5\) (see Fig. 1). In some parameter regimes the cycle is attracting, but upon addition of noise a "stochastic limit cycle" is created, that, unlike an attracting heteroclinic cycle, has a finite mean period. This is the first recorded instance of such a phenomenon, and serves as a prototype for many similar structurally stable heteroclinic cycles.

We will analyze a modified version of Eq. (1) that lacks the permutation symmetry of the axes, allowing the eigenvalue spectrum of the three fixed points located on the coordinate axes to differ. Namely,

\[
\begin{align*}
\dot{x}_1 &= x_1(q_0 + q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2), \\
\dot{x}_2 &= x_2(q_0 + q_4 x_1^2 + q_5 x_2^2 + q_6 x_3^2), \\
\dot{x}_3 &= x_3(q_0 + q_7 x_1^2 + q_8 x_2^2 + q_9 x_3^2).
\end{align*}
\]

The three on-axes fixed points will be referred to as \( p_1, p_2, p_3 \). Each of these fixed points is attracting along the coordinate axes, so that the flow is out from the origin toward each of the three fixed points. A cycle exists, as shown in Fig. 1, and if the eigenvalues of the three fixed points, \( p_i, i = 1,2,3 \), are \( \lambda_i^1, \lambda_i^2 \) (ignoring the attracting direction along the axis, which is irrelevant), the cycle is asymptotically stable if

\[
\frac{\lambda_1^1}{\lambda_2^1} \times \frac{\lambda_2^2}{\lambda_3^2} \times \frac{\lambda_3^1}{\lambda_1^1} > 1.
\]

We will be studying the case where one fixed point in the three point cycle has an unstable eigenvalue larger than the stable eigenvalue, while the whole cycle itself is attracting. We will refer to that fixed point as the unstable saddle point.

In Figs. 1 and 4 we demonstrate the difference in behavior in phase space of the cycle upon addition of very small amplitude additive noise. Figure 1 shows the cycle without added noise, drawn as a cartoon idealization in phase space. Note that a single trajectory will lie in just one octant of the phase space since it cannot cross any invariant plane. In Fig. 4(a) we see the same cycle upon the addition of Gaussian white noise of order \( 10^{-4} \). Now the trajectories lie in all eight octants of the phase space. Changing a fixed point from...
stable to unstable generates the obvious spreading of the trajectories seen in Fig. 4, at least in the 2,3 projection, and this “lift-off” prevents the crossing of the 1,2 plane. In Fig. 4c we show a cycle with two unstable saddles that demonstrates an even more marked amplification of very small amplitude noise. The lift-off away from the unstable saddles prevents the trajectory from crossing at least most of the time the 1,2 and 1,3 planes. The time series in $x_1$, $x_2$ and $x_3$ for the first two cases is shown in Figs. 2 and 3. Predicting the thickening and the mean “lift-off” of such a cycle from the coordinate axis is the subject of the next two sections.

III. REVIEW OF BASIC NOISE ANALYSIS

In this section we briefly review analysis presented in Ref. 2 and we refer the reader to that paper for further details.

The addition of small noise to an otherwise structurally stable and asymptotically stable heteroclinic cycle is modeled by splitting the cycle into local regions around the fixed points and the global connections between them. We assume the noise is small enough to be considered negligible during the global transits. This is necessarily an approximation when applied to the full nonlinear system, for two reasons. The first is that we are making a linear approximation for the dynamics, the second is that we are assuming that the noise is small enough to be neglected outside of the box around each fixed point. More exact, but much more cumbersome results can be obtained by considering the dynamics of the probability distributions of solution around the whole cycle. For examples of this kind of calculation see Ref. 3.

The neighborhoods around the fixed points are defined to be where the size of the noise is comparable to that of the vector field, or larger. That is, if the equation of motion can be written

$$\frac{dx}{dt} = f(x) + \epsilon \eta(t), \quad x \in \mathbb{R}^n,$$

with $\eta(t)$ a vector valued random process, the neighborhood of a fixed point is where $\epsilon h(t)$ and $\epsilon f(x)$ are comparable. We denote this region as $U_d$, where $d$ gives the size of a box around the fixed point. (See Fig. 5.)

To be more explicit, we write the equations for the neighborhood around a saddle point in two dimensions as

$$\begin{align*}
\frac{dx}{dt} &= -\lambda_s x dt + \epsilon dW_x, \\
\frac{dy}{dt} &= \lambda_u y dt + \epsilon dW_y,
\end{align*}$$

where $dW_x$ and $dW_y$ are zero mean independent Wiener processes, $\langle dW^2 \rangle = dt$, and $\epsilon$ is therefore the root-mean-square (r.m.s.) noise level. The stable and unstable eigenvalues are $\lambda_s > 0$ and $\lambda_u > 0$ respectively, and $x$ and $y$ are the linearized stable and unstable directions in phase space.

If $p(x,t|x_0), p(y,t|y_0)$ denote the conditional probability density functions for solutions of (5) started at $x_0, y_0$ at $t=0$, then $p$ obeys the Fokker-Planck (F.P.) equation,

$$p_t = -\lambda(zp)_z + \frac{\epsilon^2}{2} p_{zz},$$

FIG. 4. Phase plane projections of an asymptotically stable heteroclinic cycle with noise. (a) all fixed points stable, (b) fixed point $p_1$ unstable, (c) fixed points $p_1$ and $p_3$ unstable.

FIG. 5. (a) Trajectory near the heteroclinic cycle of system (2) with surfaces of section for probability distributions, (b) closeup look at one saddle point.
with \( \lambda \) replaced by \( \lambda_s \) or \( \lambda_u \) and \( z \) representing \( x \) or \( y \), for the stable and unstable process, respectively. The solution to this F.P. equation is well known, and if the initial distribution is a Gaussian, it remains a Gaussian for all time. Denoting a Gaussian distribution with mean \( \mu \) and variance \( \sigma \) as

\[
N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}},
\]

the solutions to the F.P. equation for the stable and unstable processes are

\[
p(x,t|x_0) = N\left(x_0 e^{-\lambda_u t}, \sigma_{x_0}^2 e^{-2\lambda_u t} + \frac{\epsilon^2}{2\lambda_u} (1 - e^{-2\lambda_u t})\right),
\]

\[
p(y,t|y_0) = N\left(y_0 e^{\lambda_u t}, \sigma_{y_0}^2 e^{2\lambda_u t} + \frac{\epsilon^2}{2\lambda_u} (e^{2\lambda_u t} - 1)\right).
\]

The above assumes initial distributions \( p(x,0|x_0) = N(x_0, \sigma_{x_0}) \) and \( p(y,0|y_0) = N(y_0, \sigma_{y_0}) \), i.e., mean at \((x_0, y_0)\) and variance \((\sigma_{x_0}^2, \sigma_{y_0}^2)\).

Using the above expression for the unstable process we can compute the probability distribution for the passage time through the neighborhood of the saddle point. This is a first passage time computation, which is in keeping with our assumption that the noise is small enough to be ignored outside of the neighborhood \( U_d \).

The probability that the passage time \( T \) exceeds \( t \) is the probability that a solution started at \( x = d, y = y_0 \) is still in \( U_d \) at time \( t \),

\[
P(T > t) = \int_{-d}^{d} p(y,t|y_0) dy.
\]

Let \( P(T) \) denote the probability distribution function for passage times. Then,

\[
P(T) = \frac{d}{dt}(1 - P(T > t)).
\]

Upon the change of variables \( Y(t) = y/\sigma(t); \Delta(t) = d/2\sigma(t) \) the integral for \( P(T > t) \) becomes

\[
\frac{1}{\sqrt{\pi}} \int_{-\Delta}^{\Delta} e^{-y^2} dY.
\]

So the expression (10) for \( P(T) \) can be written

\[
P(T) = \frac{d}{dt}(\text{erf}(\Delta(t)))
\]

\[
= -\frac{2}{\sqrt{\pi}} e^{-\Delta^2(t)} \Delta'(t) = \frac{2\lambda_u \Delta(t) e^{-\Delta^2(t)}}{\sqrt{\pi} (1 - e^{-2\lambda_u t})}.
\]

This expression is computed for zero mean incoming distribution, as is the case in the next sections. The analysis extends to nonzero incoming mean; this will be documented in a forthcoming paper.6

The mean passage time \( \tau = \langle T \rangle \) can be computed directly, and is estimated by

\[
\tau = \frac{1}{\lambda_u} \left( \ln\left(\frac{d}{\epsilon}\right) \text{erf}\left(\frac{d}{\epsilon\sqrt{\lambda_u}}\right) + O(1) + O(\epsilon^2) \right).
\]

In the limit as \( \epsilon \to 0 \) it can be shown2 that

\[
\tau \sim \frac{1}{\lambda_u} \left( \ln\left(\frac{d}{\epsilon}\right) + O(1) \right).
\]

For more details see Refs. 2 and 7.

We will use these results in analyzing the distribution of trajectories around the heteroclinic cycles in system (2).

IV. NOISE ANALYSIS NEAR HETEROCLINIC CYCLES

Figure 5(a) shows a trajectory near an attracting heteroclinic cycle of system (1) with surfaces of section near one saddle point to indicate the geometry of our analysis. Figure 5(b) shows a sketch of distribution functions going into and coming out of the neighborhood of the fixed point in the cycle. We will determine the shape of the incoming and outgoing distributions for each of the three fixed points of the cycle, in the case where one of the saddle points is unstable. The analysis for the case where two of the saddles are unstable while the whole cycle is asymptotically stable is an obvious extension of these results.

The time evolution of the probability distributions is governed by Eq. (6) with initial distributions

\[
p(x,t) = \delta(x - d), \quad p(y,t) = N(0, \sigma_y).
\]

We start the distribution in \( x \) at \( x = d \) because that is the condition that determines the section on which we evaluate \( p(y,t) \). The distribution in \( y \) is assumed to be normal with zero mean and some initial variance \( \sigma_y^2 \). As noted in the previous section, the solutions to (6) with these initial conditions are

\[
p(x,t) = N\left(d e^{-\lambda_u t}, \frac{\epsilon^2}{2\lambda_u} (1 - e^{-2\lambda_u t})\right),
\]

\[
p(y,t) = N\left(0, \sigma_y^2 e^{2\lambda_u t} + \frac{\epsilon^2}{2\lambda_u} (e^{2\lambda_u t} - 1)\right).
\]

The exit distribution is \( p(x,t) \) conditioned on \( y = d \), the position of the exit section. This is simply

\[
p_{\text{exit}}(x) = p(x|y = d) = \int_0^\infty p(x,t) P(t) dt,
\]

where \( P(t) \) is the probability that \( y = d \) at time \( t \). We have already computed \( P(T) \) in Sec. III, and hence the exit distribution can be written in integral form,

\[
p(x|y = d) = \int_0^\infty N\left(d e^{-\lambda_u t}, \frac{\epsilon^2}{2\lambda_u} (1 - e^{-2\lambda_u t})\right) \times \frac{2\lambda_u \Delta(t) e^{-\Delta^2(t)}}{\sqrt{\pi} (1 - e^{-2\lambda_u t})} dt.
\]

The mean of the exit distribution is \( \int_{-\infty}^\infty x p_{\text{exit}}(x) dx \), so,
FIG. 6. Distributions computed numerically from integral (16), varying the stable eigenvalue: 0.15, 0.1, 0.05, with unstable eigenvalue = 0.2. Notice that the position of the peak of the distribution increases with decreasing stable eigenvalue.

\[ \mu_{\text{exit}} = \int_{-\infty}^{\infty} x \int_{0}^{\infty} p(x,t) P(t) dt dx \]

\[ = \int_{0}^{\infty} P(t) \int_{-\infty}^{\infty} x p(x,t) dx dt = \int_{0}^{\infty} P(t) \mu(t) dt, \quad (17) \]

where the mean of \( p(x,t) \) is denoted \( \mu(t) \). We can switch the order of integration of \( x \) and \( t \) since the probability distributions in the integrand are continuous.

These integrals can be computed numerically and a few examples are plotted in Fig. 6 to show how a skewed distribution arises with decreasing stable eigenvalue. In the next section we compare these theoretical predictions for single fixed points with numerical experiments for system (2).

The scaling of the mean of the exit distribution with noise level can be estimated. Starting with the full expression

\[ \mu_{\text{exit}} = \frac{2 \lambda_s}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\frac{1}{2} \lambda_s^2 w^2} \frac{\Delta e^{-\frac{\Delta^2}{(1-e^{-\lambda_s} w)}}}{1-e^{-\lambda_s} w} dt, \quad (18) \]

upon making the substitution \( w = e^{\epsilon (2 \lambda_s^2 - 1)} \) the integral becomes

\[ \mu_{\text{exit}} = \frac{d^2 \lambda_s}{\sqrt{2 \pi \epsilon}} e^{\frac{\epsilon}{2}} \int_{0}^{\infty} \left( w + \epsilon^2 \right)^{-\alpha} \frac{\Delta e^{-\frac{\Delta^2}{(1-e^{-\lambda_s} w)}}}{w^{3/2}} e^{-\beta w} dw, \quad (19) \]

where \( \alpha = \lambda_s / 2 \lambda_u \) and \( \beta = d^2 \lambda_s / 2 \). We have followed the calculation in Stone and Holmes\(^2\) in this substitution by assuming that \( \sigma_x = 0 \). This is made valid by considering a new shifted time, since any delta function will evolve into a Gaussian under the random process considered. Including explicitly this time shift of \( 1/2 \lambda_u \log(1+\lambda_u / \lambda_s) \) does not change the scaling argument below, so for clarity in presentation we implicitly shift the time by setting \( \sigma_x = 0 \). In the limit as \( \epsilon \to 0 \), the integral is

\[ \int_{0}^{\infty} w^{-(\alpha + 3/2)} e^{-\beta w} dw, \]

which can be shown to be convergent for any value of \( \alpha \). The leading order behavior for \( \mu_{\text{exit}} \) is thus

\[ \mu_{\text{exit}} \sim e^{\epsilon^2} = e^{\lambda_s / \lambda_u}. \quad (20) \]

If \( \lambda_s > \lambda_u \) the curve comes in tangent at zero noise level, confirming our numerical intuition that the mean moves away from zero slowly as the noise is increased if the saddle point is "stable." If \( \lambda_s < \lambda_u \) the slope of the tangent to the curve at zero is infinite and small changes in noise level have a comparatively large effect on the size of the mean. This scaling law demonstrates explicitly the critical difference of the effect of noise on stable and unstable saddles.

We can compute the standard deviation for the probability distribution (16) in a similar manner and find that asymptotically

\[ \sigma_{\text{exit}} \sim c_1 e^{\lambda_s / \lambda_u}, \quad (21) \]

with constants \( c_1, c_2 \) depending on \( \lambda_s, \lambda_u, \) and \( d \). Note that for \( \lambda_s > \lambda_u \) the variance varies linearly with the noise level, whereas for \( \lambda_u > \lambda_s \) it has the same asymptotic behavior as the mean. Figure 7 shows a plot of \( \mu_{\text{exit}} - \sigma_{\text{exit}} \) for \( \lambda_s = 0.2 \), \( d = 0.1 \) and \( \epsilon = 10^{-6} \) for \( \lambda_u \) between 0.05 and 1. We see that for a "stable" saddle (\( \lambda_s > 0.2 \)), since the mean is smaller than the standard deviation, there is a significant portion of the exit distribution on both sides of the invariant plane \( x = 0 \). For an "unstable" saddle the mean is greater than \( \sigma_{\text{exit}} \); "liftoff" is seen in the phase plane, and consequently the probability of exiting with \( x < 0 \) is much reduced.

In the next section we compare these asymptotic results with those from numerical experiments.

V. EXPERIMENTAL RESULTS

We performed numerical simulations of the equations to confirm our analysis by using a noise-adapted second order Runge-Kutta scheme, or Heun method.\(^8\) Unless otherwise indicated, the noise level was set to \( 10^{-6} \), and the box size to
Initial runs with two-dimensional saddle points (no nonlinear terms) confirmed the accuracy of our calculations in Sec. IV.

Additionally, we performed numerical simulations of the full nonlinear system (2). Defining the ratio of the stable and unstable eigenvalues at the saddle points as \( \rho_i = \lambda_i^s / \lambda_i^u \), we studied the case where \( \rho_1 = 1/2, \rho_2 = 5/2, \rho_3 = 3/2 \). Therefore the first saddle point is unstable, while the other two are stable.

To analyze the flow of the distribution of solutions around the cycle we take each fixed point in turn, computing numerical incoming and outgoing distributions to and from that fixed point. We were interested in the time evolution of one-dimensional incoming probability distributions as they flow past the saddle as shown in Fig. 5. The relevant distributions are

- (i) fixed point \( p_1: p(x_2) \) for the incoming distribution and \( p(x_3) \) for the outgoing distribution;
- (ii) fixed point \( p_2: p(x_3) \) for the incoming distribution and \( p(x_1) \) for the outgoing distribution;
- (iii) fixed point \( p_3: p(x_1) \) for the incoming distribution and \( p(x_2) \) for the outgoing distribution.

Since the linear processes near the fixed points, and the noise added to each variable, are uncoupled, we can consider the two-dimensional probability distribution restricted to the dimension of interest without loss of necessary information. See Fig. 8 for the experimental distributions, each computed with 50,000 samples, and smoothed with a running three point average. We start the analysis of the flow of probability distributions along the heteroclinic cycle with fixed point \( p_3 \). Because the fixed point \( p_3 \) is strongly attracting (\( \rho_3 = 5/2 \),

\[ \text{FIG. 8. Numerical incoming and outgoing distributions, compared with theoretical predictions. See text for more detail.} \]
the exit or outgoing distribution in \( x_2 \) is strongly concentrated at 0, and therefore can be approximated by a \( \delta \)-function at zero. The effect of the noise added to the orbits, and of the global flow from \( p_3 \) to \( p_1 \) is to widen this \( \delta \)-function into a bell-shaped curve. To approximate this incoming distribution to \( p_1 \) as a Gaussian seems reasonable upon consideration of the Central Limit Theorem and the additive nature of the noise. [See Fig. 8(f)] A Wilcoxon goodness-of-fit test on this data set \((N=1000)\) yields a \( p \)-value of 0.27, which indicates that the null hypothesis of normality is not significantly violated. We use this Gaussian approximation as an incoming distribution to calculate the outgoing distribution using Eq. (16). The result is plotted in Fig. 8(a), along with the experimental distribution.

The incoming distribution to \( p_2 \) was approximated from the outgoing distribution to \( p_1 \), with a linear rescaling of the independent variable \( x_1 \), which is a map of the effect of the global flow from one saddle point neighborhood to the next. This approximation ignores the effect of the small noise outside the box, and it is not perfect, as can be seen in Fig. 8(b) where it is shown together with the numerical distribution. However, the most severe approximation we use is to approximate this incoming skewed distribution at saddle \( p_2 \) with a Gaussian with the same mean and variance. Equation (16) then yields an outgoing distribution that is tent shaped and centered around zero [Fig. 8(c)]. Since this saddle is strongly attracting, the result is insensitive to the missing higher order moments in the incoming distribution; i.e., any skewness present in the initial distribution is relatively unimportant in determining the outgoing distribution.

We again fit the incoming distribution to \( p_3 \) with a Gaussian [Fig. 8(d)]. The narrow outgoing distribution [Fig. 8(e)] confirms our initial approximations and completes the cycle.

The dependence of the mean of the exit distribution at fixed point \( p_1 \) was computed while varying noise level and unstable eigenvalue. The results of these computations are summarized in Fig. 9, where we show a log-log plot of the mean versus the noise level for varying values of \( \lambda_u \). The points are from numerical results of the full simulations, and the lines are computed from the scaling result (20). The slope for each line is taken to be \( \lambda_s / \lambda_{us} \), and the fit performed merely by changing the \( y \)-intercept. The magnitude of the \( y \)-intercept was \( O(1) \) in all three cases, which reassures us that the scaling is applicable over the range of eigenvalues tested.

VI. CONCLUSIONS

Sensitivity to noise in dynamical systems used to model and predict physical phenomena is problematic. Cataloging and understanding what sorts of systems amplify small amplitude noise to produce \( O(1) \) effects is necessary to avoid spurious behavior in model systems. In this paper we documented the extension of some results in Ref. 2 on the effect of small amplitude noise on heteroclinic cycles. We found that cycles that were asymptotically stable, regardless of the exact ratios of eigenvalues for the individual saddle points, could be understood in terms of a simple linear stochastic theory. We have predicted the "thickening" of the cycle by small amplitude noise, in particular its dependence on noise level and eigenvalues of the relevant fixed points.

One visible effect of small noise on these cycles is the jumping of trajectories over invariant subspaces. We found that whenever a saddle point associated with a heteroclinic cycle is a stable saddle \((\lambda_s > \lambda_u)\) there is a significant chance that trajectories will jump over an invariant plane. If \( \lambda_s > \lambda_u \), the probability of jumping, while still nonzero, becomes so small that jumping is rarely seen in our finite time simulations. As a result, the system that we have studied shows four qualitatively different cases: (i) if all saddles are stable, trajectories explore all eight octants of the three-dimensional phase space; (ii) if saddle \#1 is unstable \((\lambda_1^u / \lambda_1^s > 1)\) and the following saddle is so strongly stable that \( \lambda_2^s / \lambda_2^u \times \lambda_3^s / \lambda_3^u < 1 \) then the trajectories explore four octants; (iii) if saddle \#1 is strongly unstable \((\lambda_1^u / \lambda_1^s < 1)\) such that \( \lambda_2^s / \lambda_2^u \times \lambda_3^s / \lambda_3^u > 1 \) then the trajectories explore two octants; (iv) if two saddles are unstable then the trajectories also explore just two octants. Recall that we always assume that the full heteroclinic cycle is asymptotically stable, i.e., \( \lambda_2^u / \lambda_3^s \leq 1 \). With its average period and average position in phase space, has to be modified since the actual trajectories randomly switch between two, four, or eight different statistical limit cycles.

Another \( O(1) \) effect of small amplitude noise on a cycle with at least one unstable saddle is the spreading of trajectories in the neighborhood of the saddle. In a cycle with no unstable saddles the spreading scales directly with the noise amplitude, while the unstable saddle amplifies the effect of noise, creating the thickening that appears in phase plane projections. We expect this sort of thickening in cycles with saddles that have two-dimensional unstable manifolds, but for saddles with one-dimensional manifolds it only occurs when there is a specific eigenvalue spectrum of all the saddles in the cycle.

Future work will focus on extending this theory to include heteroclinic cycles with saddle points that have two-
dimensional unstable manifolds. In addition, we will study the influence of small noise on heteroclinic cycles that connect periodic orbits to each other or to other saddle points.

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6 E. Stone, preprint.