Introduction to Pattern Formation Through Plate Buckling

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When applying force to a solid body deformation can occur.

- Suppose every point is described by a position vector \( \mathbf{p} \) before deformation and \( \mathbf{p}' \) after (with components \( x_i \) and \( x'_i \)).
- Define \( u_i = x'_i - x_i \) as the displacement vector.
- Before deformation the distance between two points is \( dx_i \). After deformation, the distance \( dx'_i = dx_i + du_i \).

(Note that the summation is implied when an index is specified.)
Note also the following:

- Then the distance between the points is $d'l = \sqrt{x_i^2}$ and after deformation is $d'l' = \sqrt{x_i'^2}$.

- We can use the substitution $du_i = \frac{\partial u_i}{\partial x_k} dx_k$.

Then we have

$$d'l'^2 = d'l^2 + 2 \frac{\partial u_i}{\partial x_k} dx_k x_i + \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_l} dx_k dx_l$$
Strain Tensor

We can switch the indices in the second term on the right and interchange the $i$ and $l$ in the third term to simplify the expression for $dl'^2$:

$$dl'^2 = dl^2 + 2u_{ik}dx_idx_k$$

Where $u_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_i} \frac{\partial u_l}{\partial x_k} \right)$ is the strain tensor.

- For small deformations, the third term is negligible and so we get

$$u_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)$$

(Also note that the change in volume can be found, $dV' = dV(1 + u_{ii})$)
What is Strain

Consider the following:

\[ \frac{\ell}{\ell_0} \]

- Stretch is the ratio of the deformed length to the original length (i.e. \( \frac{\ell}{\ell_0} \))
- Any function of stretch is considered to be "strain," and it is characterized by the strain tensor.

Figure 26  Elongation of a thin rod
Elasticity

Stress is the amount of force applied to an area. When the force exceeds what the material can take, it deforms.

- This permanent deformation under stress is called plastic deformation.
- When the stress is not enough to cause permanent deformation to occur, it is called elastic deformation.
In general, there are three types of stress:

- **Shear stress**, where a force is applied parallel to the surface.
- **Tensile stress**, where a force pulls outward on the surface.
- **Compressive stress**, when a force pushes the surface inward.
When a body is not deformed, it is in a state of equilibrium with the net force on the body being zero.

- Deformation causes forces to arise in the body which "push back." This is called Internal Stress. (Newton’s 3rd law)
- This force arises from the interactions between molecules, and hence have a very short range.
- Hence the forces exerted on any part by its surroundings of the body act only on the surface of that part.
The total force on some portion of the body is given by $\int \mathbf{F} dV$. 

- $\mathbf{F}$ is the force per unit volume
- $\mathbf{F} dV$ is the force on volume element $V$. 

Then because of the third fact above, we can write this as a surface integral.

$$\int \mathbf{F}_i dV = \oint \sigma_{ik} df_k$$
The stress tensor, $\sigma_{ik}$, is the $i$th component of the force perpendicular to the $x_k$ axis.
Uniform Stress

Consider an example where pressure is applied uniformly to a body.

- This pressure, \( p \), is directed inward and normal to the surface at all points.
- A force \( p \, df_i \) acts on the surface element \( df_i \).
- Thus in terms of the stress tensor, this force is \( \sigma_{ik} \, df_k \).

Hence,

\[
\sigma_{ik} = -p \delta_{ik}
\]
Consider a deformed body where the displacement vector $u_i$ is changed by a small amount $\delta u_i$. We will determine the work done by the internal stresses.

- Multiply the force $F_i = \frac{\partial \sigma_{ik}}{\partial x_k}$ by $\delta u_i$.
- Denote $\delta R$ as the work done by internal stresses per unit volume.
- Integrating gives $\int \delta R \, dV = \int \frac{\partial \sigma_{ik}}{\partial x_k} \delta u_i$
- Then integrating by parts and making the simplifying assumption that this is an infinite medium gives the work in terms of the change in the strain tensor

$$\delta R = -\sigma_{ik} \delta u_{ik}$$
Thermodynamics - II

The first law of thermodynamics states that change in energy is heat minus work:

$$\Delta E = Q - W$$

This implies that $d\mathcal{E} = TdS - dR = TdS + \sigma_{ik} du_{ik}$.

- $d\mathcal{E}$ is an infinitesimal change in the total energy.
- $T$ is the temperature.
- $dS$ is the infinitesimal entropy.
- $dR$ is the work done by internal stresses.
Using the previous relations $dV' = dV(1 + u_{ii})$ and $\sigma_{ik} = -p\delta_{ik}$, we can write the following inequality in the setting of uniform (or hydrostatic) compression.

$$dR = \sigma_{ik}du_{ik} = -p\delta_{ik}du_{ik} = -pd_{ii} = -pdV$$

Thus we arrive at the Fundamental Thermodynamics Relation:

$$d\mathcal{E} = TdS - pdV$$
Hooke’s Law

Recall the one-dimensional form of Hooke’s law:

\[ F = -k \cdot x \]

Where \( k \) is a constant depending on the material and \( x \) is the displacement.

The general form is:

\[ \sigma_{ij} = C_{ijkl} \cdot u_{kl} \]

Where \( C_{ijkl} \) is a fourth order tensor, depending on the physical system.
In a setting where Hooke’s law holds, Young’s modulus is the measure of the stiffness in a material.

\[ E = \frac{\sigma_{ii}}{u_{ii}} \]

This is defined when both forces are acting in the same axial direction.
Poisson’s ratio

Poisson’s ratio is the ratio between the transverse strain and axial strain.

\[ \nu = \frac{u_{jj}}{u_{ii}} \]

(Note there is no summation here)
The model we are working with is one described by Michael Kücken, is based on the idea of *buckling* in the basal layer.
The basal layer is a thin, single layer made up primarily of keratinocytes, a durable protein.

It is structurally supported by hemidesmosomes, which is a protein sheet sitting between it and the epidermis.

Hence, the basal layer is a thin, elastic sheet which can resist forces and bending.
Basal Layer - II

- Cells in the basal layer are thought to reproduce faster than its surroundings.
- This leads to a compressive force on the basal layer, which can be modeled by treating the epidermis and dermis as "springs."
- Since the Compressional stress on the basal layer is greater than all of the other confining stress applied on it, buckling occurs.
Sources

Hjelmstad, Keith D. Fundamentals of structural mechanics. Springer, 2004


http://en.wikipedia.org/wiki/Tensor

http://en.wikipedia.org/wiki/Poisson’s_ratio