Symmetrically Pairwise-Bargained Allocations in an Assignment Market*

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Consider an assignment game with transferable utility where the optimally assigned partners engage in bargaining of the sort modelled by Nash, using as their threats the maximum they could receive in an alternative match. A symmetrically pairwise-bargained (SPB) allocation is a core allocation such that all partners are in bargained equilibrium. It is shown that an SPB allocation always exists, that the set of SPB allocations coincides with the intersection of the kernel and the core, and that there is a rebargaining process which converges to an SPB allocation if it begins at a “distinguished point” in the core. Journal of Economic Literature Classification Numbers: 022, 026. ©1984 Academic Press, Inc.

1. INTRODUCTION

Assignment problems abound in the world around us. Examples of such problems include the allocation of apartments to tenants, the placement of medical interns and residents at hospitals, the assignment of workers to jobs or students to colleges, and the pairing of men and women in marriage.

Two approaches have been taken in studying assignment problems. The first assumes that each agent in both disjoint groups can rank the members of the opposite group in terms of their desirability as partners in a match. Gale and Shapley [3] have shown that when each person is matched with at most one member of the opposite group, stable assignments can always be found; that is, there exist assignments such that no two individuals would both prefer to be matched with each other rather than their assigned mates. Such a stable assignment is not usually unique. Roth [7] has generalized these results to situations where members of either group may be matched with more than one member of the other group.

The other approach to assignment problems, which will be taken here,

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assumes that when a member of one group is paired with a member of the other group, additional “value” is created which must be divided between the two individuals. For example: a worker at a job produces output which must be divided between the worker’s wage and the firm’s profit, when a tenant rents an apartment, the difference between what the apartment is “worth” to the tenant and the minimum that would be accepted by the landlord is a “surplus” which must be divided between the tenant and the landlord by determining a rental price; finally, there are gains from marriage due to the division of labor within households and the concomitant investments in specialized human capital which must be divided between the spouses. Thus, we can see that prices, as well as assignments, must be determined when this second approach is taken to the assignment problem.

Standard market analyses assumes that the good (or individuals) being traded are homogeneous; that is, it does not matter which unit of the good a buyer receives. All apartments or workers or husbands are the same, and anyone willing and able to pay the price will be “assigned” one. The essential feature of an assignment problem is that the goods traded are heterogeneous, and furthermore, that they are indivisible. Thus, standard supply and demand analysis is inadequate for analyzing this particular kind of market.

Consider, for example, the marriage market. Since men and women are heterogeneous, the amount of “value,” or income, produced in each possible marriage between one woman and one man is typically different from the amount produced in every other such marriage. This being the case, two obvious questions arise. First, how will the women and men in a society pair off in marriage; and second, how will the output of the marriages be divided between the spouses?

The first of these questions, which is not completely independent of the second, can be answered by formulating this assignment problem as a sidepayment game in which the only relevant coalitions are the single-person households and the two-person marriages.\(^1\) Formulation of the problem as a sidepayment game implies that there is transferable utility among all agents in the market. An assignment is a pairing of women and men into marriages and an allocation is a vector of incomes, given some assignment, such that the sum of the incomes of the spouses in each marriage is equal to the output of that marriage. An allocation is stable if no individual could produce alone more than what s/he receives in this allocation and if no man and woman

\(^1\) It is assumed here that pairing two individuals from the same group produces no additional value. Furthermore, coalitions containing members of each group and consisting of more than two individuals produce no more value than the sum of what they can produce in pairs. In the marriage market example, then, only monogamous, heterosexual marriages produce additional value.
could form another marriage and produce together more than the sum of their incomes.

Shapley and Shubik [9] show that only under an assignment of men to women such that society’s output is maximized can a stable allocation be found. In fact, formulated in terms of linear programming, the set of solutions to the dual of the problem of maximizing society’s output is exactly that set of income vectors which are stable allocations in the sidepayment game. An assignment of men to woman which maximizes society’s output is an optimal one. The optimal assignment is generically unique.²

The core of an assignment game is the set of all allocations which are stable. Every element in the core is consistent with the optimal assignment(s) of mates. Geometrically, the core of a assignment game is a closed, convex polyhedron whose dimension is equal to at most the minimum of the number of members in one group or in the other. The core also tends to be “elongated,” with the long axis going from the point where every member of one group gets at least as much as he can in any stable allocation to the point where every member of the other group gets at least as much as she can in any stable allocation [9, p. 120]. Thus, within the core of a marriage market, the fortunes of the women tend to rise and fall together as do the fortunes of the men.

Although the optimal assignment is usually unique when the assignment problem is formulated as a sidepayment game, the core of an assignment market very rarely consists of a single point. Thus, there is a range of prices for each unit of the good that results in market stability. Are some prices more likely to occur than others? This is the question with which this paper is concerned.

Considering the individual optimal marriage in the example at hand, it can be argued that the optimally assigned individuals will engage in implicit or explicit bargaining to determine how the output of their marriage will be distributed between them. Since transferable utility is being assumed, both the symmetric Nash bargaining solution and the (symmetric) Kalai–Smorodinsky solution result in the spouses splitting equally the excess of their total income less the sum of their threats.³ Thus, as long as the threats of both spouses are known, this symmetric bargaining solution yields a unique answer to the question of how income in any individual marriage will be divided.

The threats need to be determined, then, in order to determine the bargained outcome. In a marriage formed within a marriage market, it is clear that the threat, or disagreement outcome, of each spouse is at least as

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² If the assignment game was such that there was more than one optimal assignment, a slight perturbation of the values of the two-person coalitions would result in a unique optimal assignment.

much as s/he could attain alone, that is in the single state. However, it may be more than this amount if another (potential) marriage exists in which the individual could receive an income greater than that received in the single state. But how could any individual determine what his/her income in another marriage would be? Intuitively, it seems that the income in a potential marriage would have to be determined by bargaining in that marriage as well. But that bargained income would also have to be based on threats, and again, those threats may be the income in yet another marriage rather than in the single state. Hence, it is not clear which potential marriage would offer an individual the greatest income outside of his/her present marriage, let alone what that income would be. The problem of specifying a threat based on another marriage appears to be indeterminate.\footnote{In Crawford and Rochford's [2] model a specification of threats different than the one given below which are based on bargaining in alternative matches is given.}

In order to make this problem determinate, two simplifying assumptions will be made. First, each individual knows both how much could be produced in every marriage in which s/he might participate and the present incomes of all of her/his potential marriage partners. Second, the spouses in any one actual marriage consider the incomes of all their other potential partners to be fixed. Given these two assumptions, the individual trying to determine her threat could then calculate how much income she could obtain in another marriage to be the output left over from the joint product of that marriage after the potential partner is paid his given income. Hence, her threat would be the largest of the income she could obtain in another marriage according to this calculation, or her income in the single state. This threat, then, is feasible and reflects the other market opportunities available to her.

Once both spouses compute their threats in this manner, bargaining can be used to determine how the output of their marriage will be divided. However, this outcome is only a partial equilibrium solution in the assignment market since it depends crucially on the incomes of all the other market participants. A general equilibrium solution consistent with this would be an allocation of income which is the symmetrically bargained outcome of every optimal marriage, where all threats are determined as described above. Such an allocation will be called a symmetrically pairwise-bargained (SPB) allocation.

In the next section, an example of a marriage market comprised of two women and two men is given to illustrate the assignment problem and an SPB allocation. Then, after formulating the assignment problem mathematically in Section 3, it is shown in the following section that an SPB allocation within the core of an assignment game always exists and that beginning at one of the endpoints of the "long axis" of the core, a process of
re bargaining between the optimally paired individuals will converge to an SPB allocation. The last theorem of that section shows that the set of SPB allocations coincides with the intersection of the core and the kernel of the game. The importance of these results is discussed in Section 5.

2. An Example

Suppose a marriage market is comprised of two women and two men. Normalizing, such that the income of any individual in the single state is zero, let the matrix in Table I represent the output of each of the four possible marriages. That is, \( f_1 \) and \( m_1 \) can produce 6 units of output when married to each other, \( f_1 \) and \( m_2 \) can produce 8 units, and so forth. The optimal assignment is \( f_1 \) with \( m_1 \) and \( f_2 \) with \( m_2 \). The question which remains is how will income be divided within each of the optimal marriages?

Suppose that income is divided equally between the spouses in each marriage. Then \( f_1 \) and \( m_2 \) could both be made better off by divorcing their present spouses and marrying each other. Hence, this allocation is not in the core. If the first couple were to split their output evenly, stability requires the income of \( m_2 \) to be at least 5 units, and no more than 6 units, with \( f_2 \) receiving the remainder from the 6 units. But, what would be the second couple’s symmetrically bargained incomes?

To determine this, their threats must first be calculated. The only other possible marriage for \( f_2 \) is that with \( m_1 \) in which they can produce only 2 units of output. Since \( m_1 \)’s current income is 3, if \( f_2 \) married \( m_1 \) she would have to pay him 1 unit out of her own pocket as well as giving him all of the output of the marriage in order for him to maintain his current income. Alternatively, being single would result in her income being zero. Thus, zero is the best threat that \( f_2 \) has. On the other hand, \( m_2 \)’s alternative marriage is that with \( f_1 \), in which they can produce 8 units of output. Giving \( f_1 \) the 3 units she is currently receiving would leave \( m_2 \) with 5 units if they were to marry. Thus, his threat is 5 units. The sum of the threats of \( f_2 \) and \( m_2 \) is 5, leaving 1 unit to split between them. The symmetrically bargained outcome

<table>
<thead>
<tr>
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<th>( f_1 )</th>
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<tr>
<td>( m_1 )</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>( m_2 )</td>
<td>8</td>
<td>6</td>
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in the marriage between $f_2$ and $m_2$, given that $f_1$ and $m_1$ both receive 3 units of income, is that $f_2$ will receive $\frac{1}{2}$ unit of income and $m_2$ will receive $5\frac{1}{2}$.

When $f_2$ and $m_2$ have incomes of $\frac{1}{2}$ and $5\frac{1}{2}$, respectively, these two are in bargained equilibrium; however, the first couple is not in bargained equilibrium with incomes of 3 and 3. Is there an allocation such that both couples are in bargained equilibrium simultaneously? It is easily verified that when the distribution of income $x = (x_{f_1}, x_{m_1}, x_{f_2}, x_{m_2}) = (4, 2, 1, 5)$ such a simultaneous bargained equilibrium exists. This is an SPB allocation.

3. A Mathematical Formulation

Continuing with the example of a marriage market, let us cast the assignment problem in game-theoretic form. To begin, the following definition are needed.

- $F$ is the set of female players in the marriage market, where $f \in F$, and
- $M$ is the set of male players in the marriage market, where $m \in M$.

For analytical convenience dummy players are introduced to make the number of female players and the number of male players in the marriage market equal. If a player is "married" to a dummy, that player is, in effect, single. In addition to the dummy players added to the set with the least number of actual players, another dummy player is introduced into each set. The need for this player will become apparent below. Hence, $\#F = \#M = n$ is one more than the largest of the number of actual women or the number of actual men in the marriage market. Furthermore,

- $N := F \cup M$, and $\#N = 2n$;
- $S \subset N$ is a coalition of players in $N$.
- $\mathcal{N} := \{S \mid S \subset N, S \neq \emptyset\}$ is the set of all non-empty coalitions, and $\#N = 2^n - 1$.
- $F_S := F \cap S$ is the set of all female players in $S$.
- $M_S := M \cap S$ is the set of all male players in $S$.

We can define the set of all one-to-one mappings from the women in a coalition to the men in that coalition as

- $I(F_S, M_S) := \{h : F_S \rightarrow M_S \mid h \text{ is one-to-one}\}$.

Notice that if $\#F_S > \#M_S$, $I(F_S, M_S) = \emptyset$. We can also define the set of all
one-to-one mappings from the men in a coalition to the women in that coalition as

\[ I(M_S, F_S) := \{ g : M_S \to F_S \mid g \text{ is one-to-one} \}. \]

If \( \#M_S > \#F_S \), \( I(M_S, F_S) = \emptyset \). When \( \#F_S = \#M_S \), if \( h \in I(F_S, M_S) \), then \( h^{-1} \in I(M_S, F_S) \); and if \( g \in I(M_S, F_S) \), then \( g^{-1} \in I(F_S, M_S) \). In particular, when \( S = N \), if \( h \in I(F, M) \) then \( h^{-1} \in I(M, F) \), and if \( g \in I(M, F) \) then \( g^{-1} \in I(F, M) \). A mapping \( h \in I(F, M) \) represents an assignment, or pairing, of men and women. \( \#I(F, M) = n! \).

We are now in a position to define the characteristic function of the marriage market game, \( v : \mathcal{N} \to R \), which gives the payoff, measured in units of transferable utility, of each coalition. This payoff denotes the amount of income the coalition can produce in excess of the sum of what the members of the coalition could produce on their own, regardless of what the players outside of the coalition do. (In other words, \( v \) is a zero-normalized game.) Assuming that only monogamous, heterosexual marriages produce any excess for their members, the characteristic function can be completely characterized by an \( n \times n \) matrix of payoffs from the marriages of each woman to each man. Each element of the matrix must be non-negative since the least that two people could produce together is the sum of what they could produce alone. Hence, \( v(S) \geq 0 \ \forall S \) such that \( \#F_S = \#M_S = 1 \). Furthermore, at least one row and one column of the matrix will consist of all zeroes since both sets \( F \) and \( M \) contain at least one dummy. For all coalitions, \( S \),

\[
v(S) = \begin{cases} 
0, & \text{if } \#S = 1 \\
0, & \text{if } F_S = \emptyset \text{ or } M_S = \emptyset \\
\max_{h \in I(F_S, M_S)} \sum_{f \in F_S} v(f, h(f)), & \text{if } \#F_S \leq \#M_S \\
\max_{g \in I(M_S, F_S)} \sum_{m \in M_S} v(m, g(m)), & \text{if } \#M_S \leq \#F_S.
\end{cases}
\]

Finally, the maximum payoff to society from the game is the largest amount which can be produced by the members of society, when all players pair off in marriages. That is,

\[
v(N) = \max_{h \in I(F, M)} \sum_{f \in F} v(f, h(f)).
\]

The mapping which maximizes society's output will be called the optimal

\[^3\] The payoff to coalitions consisting of distinct members \( i, j \in N \) will be written \( v(i, j) \) rather than the more cumbersome \( v((i, j)) \).
assignment and will be denoted by $h^*$ (or $g^*$). When the mapping which maximizes society's output is not unique, $h^*$ will denote any arbitrarily chosen optimal assignment. This mapping represents the assignment of spouses, or set of marriages, which corresponds to the set of stable income vectors. The marriages which correspond to the optimal assignment will be called the optimal marriages.

Define an allocation to be a non-negative vector $x = ((x_i)_i \in I, \ (x_m)_{m \in M}) \in R_+^N$ such that $\sum_{i \in N} x_i = v(N)$, denoting the income of every player. The set of all such allocations is known as the imputation space. Because allocations are non-negative, every allocation is individually rational. Given any allocation $x$, we can define a function $x: \mathcal{P} \to R$ such that

$$x(S) = \sum_{i \in S} x_i.$$  

We can also define the excess of $S$ at $x$ as

$$e(S, x) = v(S) - x(S).$$

The core of the game can now be defined as

$$C(v) := \{x \in R_+^n \mid e(S, x) \leq 0, \forall S \in \mathcal{P}, \text{ and } e(N, x) = 0\}.$$  

This means that for the marriage market game $v, x \subset C(v)$ if and only if (iff)

$$x_i \geq 0 \quad \forall i \in N $$

$$x_f + x_{h^*(f)} = v(f, h^*(f)) \quad \forall f \in F$$

and

$$x_f + x_{h(f)} \geq v(f, h(f)) \quad \forall f \in F \text{ and } \forall h \in I(F, M).$$

Within any of the optimal marriages, the spouses can use bargaining to determine their incomes if they know their threats. To determine the threats specified in the first section, the incomes of all the other participants in the marriage market besides these two individuals must be known. Define a partial allocation as a vector $x^0 \in R_{++}^n$ which denotes the incomes of all the other players besides $f$ and $m = h^*(f)$. Then the threats of $f$ and $m$ under this partial allocation are

$$\hat{t}_f(x^0) = \max_{\substack{m' \in M \atop m' \neq m}} (v(f, m') - x_{m'})$$

and

$$\hat{t}_m(x^0) = \max_{\substack{f' \in F \atop f' \neq f}} (v(f', m) - x_{f'}).$$
At least one of the potential marriages for each spouse is marriage to a dummy; that is, being single. Let \( m^o \in M \) and \( f^o \in F \) denote dummy players. Then, by definition, \( v(f, m^o) = v(f^o, m) = 0 \). Furthermore, under any core allocation \( x_{f^o} = x_{m^o} = 0 \). Hence,
\[
\tilde{f}(x^{(f)}) \geq 0 \quad \text{and} \quad \tilde{m}(x^{(f)}) \geq 0 \quad \forall x \in C(v).
\]
The symmetrically bargained incomes of the spouses then are
\[
\tilde{b}_f(x^{(f)}) = \tilde{f}(x^{(f)}) + \frac{1}{2}(v(f, m) - \tilde{f}(x^{(f)}) - \tilde{m}(x^{(f)}))
\]
and
\[
\tilde{b}_m(x^{(f)}) = \tilde{m}(x^{(f)}) + \frac{1}{2}(v(f, m) - \tilde{f}(x^{(f)}) - \tilde{m}(x^{(f)})).
\]

For notational convenience we can define
\[
t_f(x) = \tilde{f}(x^{(f)}),
\]
\[
t_m(x) = \tilde{m}(x^{(f)}),
\]
\[
b_f(x) = \tilde{b}_f(x^{(f)}),
\]
and
\[
b_m(x) = \tilde{b}_m(x^{(f)}).
\]

A symmetrically pairwise-bargained (SPB) allocation is a core allocation such that the output of each optimal marriage is divided between the spouses through the use of symmetric bargaining. The definition can be given precisely as follows. \( x \) is an SPB allocation iff \( x \in C(v) \), and \( \forall f \in F \), \( b_f(x) = x_f \) and \( b_m(x) = x_m \) where \( m = h^*(f) \). Finally, the set of SPB allocations is denoted as
\[
\text{SPB}(v) = \{ x \in C(v) \mid x_f = b_f(x) \text{ and } x_m = b_m(x) \text{ for } m = h^*(f), \forall f \in F \}.
\]

4. Main Results

Now that the problem has been formulated mathematically, the theorems can be stated and proved.

Theorem 1. \( \text{SPB}(v) \neq \emptyset \).

Proof. The proof of this theorem is based on the use of Brouwer's Fixed Point Theorem, and is given in four steps.

\(^a\) This theorem actually follows directly from Theorem 3, since it is well known that the core of an assignment game is non-empty and that the intersection of the kernel and the core of any game is non-empty whenever the core is non-empty. However, this proof of Theorem 1 is useful for introducing the re-bargaining process, which is used in Theorem 2.
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Step 1. First, we want to define a function whose domain is the core of the game. A point in the core represents a vector of incomes which necessarily corresponds to the optimal assignment of spouses. The optimal marriages can be arbitrarily ordered with the woman and man in the \( n \)th marriage being denoted as \( f_i \) and \( m_i \), for \( i = 1, \ldots, n \). Now we can define a function \( a: \mathcal{C}(v) \rightarrow \mathbb{R}^{2n} \) under which the components of the image vector of \( x \) are determined two at a time, and are based on the components already determined. Specifically, the components representing the incomes of the spouses of the first marriage are given to be the symmetrically bargained solution to the problem of how the output from their marriage is to be divided, where their threats are based on the partial allocation given by the vector \( x \) omitting the first two components. The next two components of the image of \( x \), representing the incomes of the spouses in the second marriage, are likewise given to be the bargained distribution of income, where the threats are based on the partial allocation given by the components of \( x \) representing the incomes of the spouses in the third and higher numbered marriages and the newly determined incomes of the spouses in the first marriage. The remaining components of the image of \( x \) are similarly determined.

Formally, let

\[
a(x) = (a_f(x), a_m(x))^n_{i=1},
\]

where

\[
a_f(x) = t_{f_i}(x) + \frac{1}{2}(v(f_i, m_i) - t_{f_i}(x) - t_{m_i}(x))
\]

and

\[
a_m(x) = t_{m_i}(x) + \frac{1}{2}(v(f_i, m_i) - t_{f_i}(x) - t_{m_i}(x)),
\]

for \( i = 1, \ldots, n \), and where the threats are

\[
t_f(x) = \max_{f < i} (v(f_i, m_j) - a_f(x)), \quad \max_{f > i} (v(f_i, m_j) - x_{f_j})
\]

and

\[
t_m(x) = \max_{f < i} (v(f_i, m_j) - a_f(x)), \quad \max_{f > i} (v(f_i, m_j) - x_j).
\]

Step 2. If \( x \in \mathcal{C}(v) \), then \( a(x) \in \mathcal{C}(v) \). This is immediate once the following is established.

For any \( k = 2, \ldots, n \), if \( (a_f(x), a_m(x))^{i=1}_{i=k-1}, (x_{f_i}, x_{m_i})^{i=k+1}_{i=n} \in \mathcal{C}(v) \) then \( (a_f(x), a_m(x))^{i=1}_{i=k}, (x_{f_i}, x_{m_i})^{i=k+1}_{i=n} \in \mathcal{C}(v) \). Let \( y = (f_{j_1}, y_{m_1})^{n}_{i=1} = (a_f(x), a_m(x))^{i=1}_{i=k-1}, (x_{f_i}, x_{m_i})^{i=k+1}_{i=n} \in \mathcal{C}(v) \). Then,
where * indicates the maximizer. Because \( y \in C(v) \), the first term after the last equality sign is equal to zero and each of the other two terms are non-negative. Hence,

\[
v(f_k, m_k) - t_{f_k}(y) - t_{m_k}(y) \geq 0,
\]

and

\[
a_{f_k}(x) = t_{f_k}(y) + \frac{1}{2}(v(f_k, m_k) - t_{f_k}(y) - t_{m_k}(y)) \geq t_{f_k}(y)
\]

and

\[
a_{m_k}(x) = t_{m_k}(y) + \frac{1}{2}(v(f_k, m_k) - t_{f_k}(y) - t_{m_k}(y)) \geq t_{m_k}(y).
\]

Since \( a_{f_k}(x) \geq t_{f_k}(y) \) and \( a_{m_k}(x) \geq t_{m_k}(y) \), then

\[
a_{f_k}(x) \geq v(f_k, m_i) - y_{m_i}, \quad \text{or} \quad a_{f_k}(x) + y_{m_i} \geq v(f_k, m_i), \quad \forall i \neq k,
\]

and

\[
a_{m_k}(x) \geq v(f_i, m_k) - y_{f_i}, \quad \text{or} \quad a_{m_k}(x) + y_{f_i} \geq v(f_i, m_k), \quad \forall i \neq k.
\]

Since \( a_{f_k}(x) + a_{m_k}(x) = v(f_k, m_k) \), then \(((a_{f_k}(x), a_{m_k}(x)))_{i=1}^{k}, (x_{f_i}, x_{m_i})_{i=k+1}^{n} \in C(v)\).

By a similar argument it can be shown that if \((x_{f_i}, x_{m_i})_{i=1}^{k-1} \in C(v)\) then \(((a_{f_i}(x), a_{m_i}(x)), (x_{f_i}, x_{m_i})_{i=k+1}^{n}) \in C(v)\). Hence, by induction we can see if \(x = (x_{f_i}, x_{m_i})_{i=1}^{n} \in C(v)\), then \((x) = (a_{f_i}(x), a_{m_i}(x))_{i=1}^{n} \in C(v)\). So the range, as well as the domain, of the function is the core of the game \( v \).

**Step 3.** The function \( a \) is a continuous function of \( x \). This is immediate once the following is established. \( \forall k = 2, \ldots, n \), if \( a_{f_i}(x) \) and \( a_{m_i}(x) \) are continuous in \( x \) for \( i = 1, \ldots, k - 1 \), then \( a_{f_i}(x) \) and \( a_{m_i}(x) \) are continuous in \( x \). By definition

\[
t_{f_k}(x) = \max_{j < k} (v(f_j, m_j) - a_{f_j}(x)), \max_{j > k} (v(f_j, m_j) - x_j)
\]

and

\[
t_{m_k}(x) = \max_{j < k} (v(f_j, m_k) - a_{f_j}(x)), \max_{j > k} (v(f_j, m_k) - x_j).
\]
Each term over which the maximum is being taken is continuous in \( x \), so the maximum is continuous in \( x \). Since \( a_{ij}(x) \) and \( a_{mj}(x) \) are linear functions of \( t'_{ij}(x) \) and \( t'_{mj}(x) \), which are continuous, \( a_{ij}(x) \) and \( a_{mj}(x) \) are continuous for \( k = 2, \ldots, n \).

By a similar argument it can be shown that \( a_{ij}(x) \) and \( a_{mj}(x) \) are continuous in \( x \). Hence, by induction it is clear that each component of \( a(x) \) is continuous in \( x \), so \( a(x) \) is continuous in \( x \).

Step 4. Since \( a: C(v) \rightarrow C(v) \) is continuous and \( C(v) \) is a non-empty, convex, compact subset of \( R^{2n} \), by Brouwer's Fixed Point Theorem there exists some \( \bar{x} \) such that \( a(\bar{x}) = \bar{x} \). This implies that

\[
t'_{ij}(\bar{x}) = \max_{j \neq i} (v(f_j, m_j) - \bar{x}_m) = t_{ij}(\bar{x})
\]

and

\[
t'_{mj}(\bar{x}) = \max_{j \neq i} (v(f_j, m_j) - \bar{x}_i) = t_{mj}(\bar{x}).
\]

Therefore,

\[
\bar{x}_i - a_{ij}(\bar{x}) - t_{ij}(\bar{x}) = \frac{1}{2}(v(f_l, m_l) - t_{ij}(\bar{x}) - t_{mj}(\bar{x})) = b_{ij}(\bar{x})
\]

and

\[
\bar{x}_m - a_{mj}(\bar{x}) - t_{mj}(\bar{x}) = \frac{1}{2}(v(f_l, m_l) - t_{ij}(\bar{x}) - t_{mj}(\bar{x})) = b_{mj}(\bar{x})
\]

for \( l = 1, \ldots, n \). Hence, \( \bar{x} \) is a symmetrically pairwise-bargained allocation. So \( SPB(v) \neq \emptyset \).

The next theorem shows that the rebargaining process described by the function \( a \) actually converges to an SPB allocation when it begins at one of the endpoints of the long axis of the core.

**Theorem 2.** Let \( \{x^k\}_{k=0}^{\infty} \) be a sequence of points such that \( x^{k+1} = a(x^k) \). If \( x^0 \) is an endpoint of the "long axis" of \( C(v) \), then \( \{x^k\}_{k=0}^{\infty} \) converges to a point \( \bar{x} \in SPB(v) \).

**Proof.** Let \( x^0 \) be the point in the core which is the worst core allocation for all the women; that is, \( x^0_x \leq x_x \) \( \forall x \in C(v) \) and \( \forall f \in F \). In Step 1 of the proof, induction is used to show that \( x^{k+1} \geq x^k \) \( \forall k \) and \( \forall f \in F \); in Step 2, this fact is used to show that \( \{x^k\}_{k=0}^{\infty} \) converges; and in Step 3, it is shown that the point to which the sequence converges is an SPB allocation. (If \( x^0 \) is the point in the core which is the worst core allocation for all the men, it can be shown that \( x^{k+1} \geq x^k \) \( \forall k \) and \( \forall m \in M \). The rest of the proof then follows directly.)

Step 1. Since \( x^0 \in C(v) \) and the function \( a \) maps core allocations to
core allocations, $x^k \in C(v) \forall k$. By definition of $x^k$, we know that $x^k_j = a_j(x^k) \geq x^0_j \forall f \in F$. Now suppose $x^k_j \geq x^{k-1}_j \forall i$. By induction on $i$ we can show that this implies $x_i^{k+1} \geq x_i^k \forall i$. Suppose $x^{k+1}_j \geq x^k_j$ for $j = 1, \ldots, t$. Define

$$ y^{k,i} = ((x^{k+1}_j, x^{k+1}_m)_{j=1}^t, (x^k_j, x^k_m)_{m=t+1}^p) $$

Then

$$ x^{k+1}_j = a_j(x^k) = b_j(y^{k,i}) $$

$$ = t_j(y^{k,i}) + \frac{1}{2}(v(f_j, m_i) - t_m(y^{k,i}) - t_m(y^{k,i})) $$

$$ = \frac{1}{2}(t_j(y^{k,i}) + v(f_j, m_i) - t_m(y^{k,i})) $$

where

$$ t_j(y^{k,i}) = \max_{j \neq i} (v(f_j, m_i) - y^{k,i}_m) $$

and

$$ t_m(y^{k,i}) = \max_{j \neq i} (v(f_j, m_i) - y^{k,i}_j). $$

Obviously $t_m$ is a non-positive function of $(y^{k,i})_{j=1}^p$ and, since $y^{k,i}_m = v(f_j, m_i) - y^{k,i}_j$, $t_j$ is a non-negative function of $(y^{k,i})_{j=1}^p$. So $x^{k+1}_j = b_j(y^{k,i}) = a_j(x^k) \geq x^k_j \forall i$, whenever $x^{k-1}_j \leq x^k_j$. Since $x^0_i \leq x^k_i \forall f \in F$, this means that $x^{k+1}_j \geq x^k_j \forall k$ and $\forall f$.

Step 2. $C(v)$ is compact. $\forall f \in F$ this implies that $\{x_j \in R \mid x_j \in C(v)\}$ is bounded. Since $x^{k+1}_j \geq x^k_j \forall k$, by the Monotone Convergence Theorem, $\{x^k\}_{k=0}^\infty$ converges. Let $\lim_{k \to \infty} x^k = \bar{x}$. Since $x^k_m = v(f, m) - x^k_j \forall x \in C(v)$ when $m = h^*(f)$, and since $C(v)$ is a closed set, $\lim_{k \to \infty} x^k = \bar{x}$ where $\bar{x}_m = v(f, m) - \bar{x}_j$ for $m = h^*(f), \forall f \in F$.

Step 3. By Step 3 in the proof of Theorem 1, we know that the function $a$ is continuous. Therefore $\lim_{k \to \infty} x^k = \bar{x}$ implies that $\lim_{k \to \infty} a(x^k) = a(\bar{x})$. But $a(x^k) = x^{k+1}$, and obviously $\lim_{k \to \infty} x^{k+1} = \lim_{k \to \infty} x^k = \bar{x}$. Hence $a(\bar{x}) = \bar{x}$, so $\bar{x} \in SPB(v)$.

The last theorem of the paper shows the relationship between an SPB allocation and another known game-theoretic solution concept. Let $K(v)$ denote the kernel of the assignment game $v$ (with respect to the grand condition, $N$).

**Theorem 3.** $SPB(v) = C(v) \cap K(v)$. 
**Proof.** The proof of this theorem is given in four steps. Step 1 defines the kernel. In Step 2 some important equalities are derived. Step 3 shows that $C(v) \cap K(v) \subset \text{SPB}(v)$ and Step 4 shows that $\text{SPB}(v) \subset C(v) \cap K(v)$.

Step 1. To characterize the kernel of the game, the following definitions are needed. First define the set

$$T_{i,j} = \{ S \in \mathcal{S} \mid i \in S, j \notin S \}.$$

The maximum excess of $i$ over $j$ at $x$ is

$$s_{i,j}(x) = \max_{S \in T_{i,j}} e(S, x) = \max_{S \in T_{i,j}} (v(S) - x(S)).$$

Since the assignment game $v$ is a superadditive game, the kernel of $v$ is equal to the prekernel of $v$.\footnote{A superadditive game is one such that if $S, T \in \mathcal{S}$ and $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$. See Maschler et al. [4] for definitions of the kernel and prekernel, and a theorem on their equality for superadditive games.} Hence

$$K(v) = \left\{ x \in \mathbb{R}_{+}^{N} \mid \text{where } \sum_{i \in N} x_{i} = v(N) \text{ and } s_{i,j}(x) = s_{j,i}(x), \forall i, j \in N, i \neq j \right\}.$$

Step 2. For any $x \in C(v)$ it will be established that

- $s_{f,f}(x) = s_{f',f}(x) = 0 \quad \forall f, f' \in F, f \neq f'$
- $s_{m,m}(x) = s_{m',m}(x) = 0 \quad \forall m, m' \in M, m \neq m'$
- $s_{f,m}(x) = s_{m',f}(x) = 0 \quad \forall f \in F, m \in M,$
  where $m' = h^{*}(f)$ and $f = g^{*}(m')$
- $s_{f,m}(x) = -x_{f} + t_{f}(x) \quad \forall f \in F, m = h^{*}(f)$
- $s_{m,f}(x) = -x_{m} + t_{m}(x) \quad \forall m \in M, f = g^{*}(m)$

$\forall f, f' \in F$, $s_{f,f}(x) = \max_{S \in T_{f,f}} e(S, x) \leq 0$ by definition of the core. Let $S = \{ f, m \}$ where $m = h^{*}(f)$. Then $S \notin T_{f,f}$, and $x(S) = v(S)$. So $e(S, x) = 0$. Hence, $s_{f,f}(x) = 0$. Similarly, it can be shown that $s_{f',f}(x) = s_{m,m}(x) = s_{m,m}(x) = 0$. $\forall f \in F$ and $m' = h^{*}(f)$, $s_{f,m}(x) = \max_{S \in T_{f,m}} e(S, x) \leq 0$ by definition of the core. Let $S = \{ f, m \}$ where $m = h^{*}(f)$. Then $S \notin T_{f,m}$, and $x(S) = v(S)$. So $e(S, x) = 0$ and $s_{f,m}(x) = 0$. Similarly it can be shown that $s_{m',f}(x) = 0$.\footnote{A superadditive game is one such that if $S, T \in \mathcal{S}$ and $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$. See Maschler et al. [4] for definitions of the kernel and prekernel, and a theorem on their equality for superadditive games.}
Finally, by partitioning $T_{f,m}$, the last two equalities can be derived. Define the following subsets of $T_{f,m}$.

$$T^1_{f,m} = \{ S \subseteq T_{f,m} \mid M_S = \emptyset \},$$

$$T^2_{f,m} = \{ S \subseteq T_{f,m} \mid \#S = 2 \text{ and } \#M_S = 1 \},$$

$$T^3_{f,m} = \{ S \subseteq T_{f,m} \mid \#S \geq 3 \text{ and } M_S \neq \emptyset \}.$$

Then $(T^1_{f,m}, T^2_{f,m}, T^3_{f,m})$ forms partition of $T_{f,m}$. Hence, $s_{f,m}(x) = \max(\max_{S \subseteq T^1_{f,m}} e(S, x), \max_{S \subseteq T^2_{f,m}} e(S, x), \max_{S \subseteq T^3_{f,m}} e(S, x)).$

(a) It will be shown that $\max_{S \subseteq T^1_{f,m}} e(S, x) \leq -x_f + t_f(x)$. Let $x \in C(v)$. Then $x_f \geq 0$, $\forall f' \in F$. $\forall S \subseteq T^1_{f,m}$, $v(S) = 0$ and

$$e(S, x) = -\sum_{f' \in F_S} x_{f'} = -x_f - \sum_{f' \notin F_S} x_{f'} \leq -x_f \leq -x_f + t_f(x),$$

since $t_f(x)$ is always non-negative.

(b) It will be shown that $\max_{S \subseteq T^2_{f,m}} e(S, x) = -x_f + t_f(x)$. $\forall m' \neq m$

$$e([f, m'], x) = v(f, m') - x_f - x_{m'} = -x_f + v(f, m') - x_{m'}.$$

So,

$$\max_{S \subseteq T^2_{f,m}} e(S, x) = -x_f + \max_{m' \in M \backslash \{m\}} (v(f, m') - x_{m'}) = -x_f + t_f(x).$$

(c) It will be shown that $\max_{S \subseteq T^3_{f,m}} e(S, x) \leq -x_f + t_f(x)$. Let $x \in C(v)$. $\forall S \subseteq T^3_{f,m}$ such that $\#M_S \geq \#F_S$,

$$v(S) = \max_{h \in \{f, m\}^{\#F_S}} \sum_{f' \notin F_S} v(f', h(f')).$$

Let $h_S^m$ be a maximizer. Then

$$e(S, x) = \sum_{f' \notin F_S} v(f', h_S^m(f'))) - \sum_{i \in S} x_i$$

$$= v(f, h_S^m(f)) - x_f - x_{h_S^m(f')} + \sum_{f' \notin F_S} (v(f', h_S^m(f'))) x(f', h_S^m(f')))$$

$$+ \sum_{m \in M_S \backslash h_S^m(f'), \forall f' \in F_S} (-x_m).$$
Since \( x \in C(v) \), the two summation terms are non-positive. Hence
\[
e(S, x) \leq -x_f + v(f, h(f)) - x_{h(f)}(f) \leq -x_f + t_f(x).
\]

Alternatively, \( \forall S \in T_{f, m}^1 \) such that \( |M_S| < |F_S| \),
\[
v(S) = \max_{m \in M_S} \sum_{m' \in M_S} v(m', g(m')).
\]

Let \( g_S^S \) be a maximizer. Then, \( e(S, x) = \sum_{m' \in M_S} v(m', g_S^S(m')) - \sum_{i \in S} x_i \). Now, by definition of \( T_{f, m}^1, f \in S \) and either there exists some \( m_s \in M_S \) such that \( g_S^S(m_s) = f \), or there does not. If there is some such \( m_s \), then parallel to the above argument, we can conclude that
\[
e(S, x) = v(m_S, f) - x_f - x_{m_s} + \sum_{m' \in M_S} (v(m', g_S^S(m')) - x(m', g_S^S(m')))
+ \sum_{\substack{m' \in M_S \setminus \{m_s\} \in F_S \setminus \{f\}}} (-x_f)
\leq -x_f + v(m_s, f) - x_{m_s} \leq -x_f + t_f(x).
\]

If there is not such an \( m_s \), then
\[
e(S, x) = \sum_{m' \in M_S} (v(m', g_S^S(m')) - x(m', g_S^S(m'))) - x_f + \sum_{\substack{m' \in M_S \setminus \{m_s\} \in F_S \setminus \{f\}}} (-x_f).
\]

Since \( x \in C(v) \), the two summations are non-positive so, \( e(S, x) \leq -x_f \leq -x_f + t_f(x) \), since \( t_f(x) \) is always non-negative. Hence,
\[
\max_{S \in T_{f, m}^1} e(S, x) \leq -x_f + t_f(x).
\]

By (a), (b), and (c), we can conclude that if \( x \in C(v) \), then
\[
s_{f, m}(x) = -x_f + t_f(x) \quad \forall f \in F, \text{ where } m = h_f(f).
\]

By a parallel argument it can be shown that
\[
s_{m, f}(x) = -x_m + t_m(x) \quad \forall m \in M, \text{ where } f = g_m(m).
\]

Step 3. If \( x \in C(v) \cap K(v) \), then \( s_{f, m}(x) = s_{m, f}(x) \) \( \forall f \in F, \text{ where } m = h_f(f) \). Therefore, \( x \in C(v) \cap K(v) \) implies that
\[
-x_f + t_f(x) = -x_m + t_m(x).
\]
Since \( x \in C(v) \) we know that
\[
x_f + x_m = v(f, m).
\]
From these two equations we can easily derive \( \forall f \in F \) and \( m = h^*(f) \)
\[
x_f = \frac{1}{2}(v(f, m) + t_f(x) - t_m(x)) = b_f(x),
\]
\[
x_m = \frac{1}{2}(v(f, m) + t_m(x) - t_f(x)) = b_m(x).
\]
Hence if \( x \in C(v) \cap K(v) \), then \( x \in \text{SPB}(v) \).

Step 4. It is now shown that if \( x \in \text{SPB}(v) \) then \( x \in C(v) \cap K(v) \).
Since \( \text{SPB}(v) \subset C(v) \) and, by Step 2, \( s_{i,j}(x) = 0 \) for all other combinations of \( i \) and \( j \), to prove this assertion it need only be shown that for \( \forall x \in \text{SPB}(v) \),
\[
s_{f,m}(x) = s_{m,f}(x) \ \forall f \in F, \text{ where } m = h^*(f).
\]
For \( x \in \text{SPB}(v) \)
\[
x_f = \frac{1}{2}(v(f, m) + t_f(x) - t_m(x))
\]
and
\[
x_m = \frac{1}{2}(v(f, m) + t_m(x) - t_f(x)).
\]
By Step 2 and substituting for \( x_f \) and \( x_m \) from above
\[
s_{f,m}(x) = -x_f + t_f(x) = \frac{1}{2}(t_f(x) + t_m(x) - v(f, m))
\]
and
\[
s_{m,j}(x) = -x_m + t_m(x) = \frac{1}{2}(t_m(x) + t_f(x) - v(f, m)).
\]
So \( \forall x \in \text{SPB}(v) \), \( s_{f,m}(x) = s_{m,f}(x) \ \forall f \in F \) where \( m = h^*(f) \). [2]

5. DISCUSSION

The symmetrically pairwise-bargained (SPB) allocation was formulated in response to the question of how income might be distributed, or prices determined, within an assignment market. The problem was characterized as a bargaining problem between the optimally paired individuals, where their other market opportunities were reflected in their threats. In view of this, the finding that the set of SPB allocations in an assignment game is exactly the same as the intersection of the kernel and the core of the game is not too surprising, given that the kernel can be thought of as the set of income vectors such that all pairs of players are in balance [10, pp. 342–343]. In a game where the resulting coalition structure contains coalitions with more than two members, it is not clear why a solution based on pairwise equilibrium is appropriate; however, in an assignment game the
appropriateness of such an equilibrium is obvious. Furthermore, according to Shubik, "There have been no significant applications to the social sciences in which the kernel plays a central role" [10, p. 282]. Yet, this study indicates that the kernel is a particularly appropriate solution concept for an assignment market. Hence, the properties of the kernel in assignment games deserve further study.

The kernel may contain more than a single point, as may the set of SPB allocations. However, the nucleolus, a solution concept which was introduced by Schmeidler [8], is in both the kernel of the game and the core when the core is non-empty, and is unique. The nucleolus is usually thought of as a "fair" outcome in the sense that it is "the result of an arbitrator's desire to minimize the dissatisfaction of the most dissatisfied coalition (in the game)" [4, p. 303]. We now see that in an assignment game, the nucleolus is a "fair" outcome in the sense that all SPB allocations are "fair"; that is, in being a bargained outcome which, while treating the players symmetrically, reflects the strengths of the players with respect to their other market opportunities.

Choosing the nucleolus as the "fair" outcome of an assignment game contrasts with the "fair" outcome suggested by Thompson [11]. He defines his "fair division point" of an assignment game as the midpoint of the "long axis" of the core. At this fair division point each individual receives the average of the most and the least that s/he can receive under any core allocation.

Roth's [7] discussion of common and conflicting interests on the part of the participants in an assignment market provides a basis for understanding the difference between Thompson's fair division point and an SPB allocation. Viewed as an institution designed to achieve an optimal matching of, for example, women and men, an assignment market of the type we have been discussing serves to advance the common interests of the men and women wishing to be matched, and to resolve the conflicting interests of the men, who are in competition for desirable women, and of the women, who are in competition for desirable men [7, p. 1]. This overall pattern of common and conflicting interests is apparently reversed when attention is confined to the set of stable outcomes, that is, the core, since within the core there exist male-optimal and female-optimal allocations that are the endpoints of the "long axis." Thompson's fair division point reconciles the conflicting interests of the men and women over the set of core allocations by treating the two sides of the market as the two agents in a constant-sum game, represented by the "long axis" of the core, and, invoking symmetry, choosing

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Footnote: In the example in Section 2 of this paper, the set of SPB allocations is the line segment connecting the points (4, 2, 1, 5) and (5, 1, 2, 4) where \( x = (x_1, x_2, x_3, x_4) \). SPB(x) for all two women/two men marriage markets is derived in Roth and Desai [8].
its midpoint. On the other hand, an SPB allocation reconciles the conflicting interests of the men and women by reconciling the conflicting interests of each optimally paired man and woman in their individual marriage, or, since there is transferable utility, their constant-sum game. Under an SPB allocation, each optimally paired man and woman receives the amount of their respective threats, which represents their other market opportunities, and invoking symmetry, divide the excess from their marriage equally. Thus, rather than treating the sides of the market symmetrically as Thompson does, here the individuals are all treated symmetrically. Obviously, this results in a different type of “fairness.”

The second theorem of this paper is concerned with the properties of the rebargaining process rather than the properties of an SPB allocation itself. The importance of this result, that the rebargaining process converges to an SPB allocation starting from either endpoint of the “long axis” of the core, is perhaps best seen in light of the model of an assignment market formulated by Crawford and Knoer [1]. Their study contains the only description known to this author of a process by which the participants in an assignment market, through their own actions, will end up optimally matched and at a core allocation. This process starts with each member on one side of the market making an offer to an agent on the other side, while every individual on the receiving side keeps the best offer on hold, rejecting all others. Each player whose offer has been rejected then makes an offer to a different player on the opposite side or increases the amount of his offer to the individual to whom he made his previous offer. As long as increases in offers are made in some given discrete unit, the optimal assignment and a core allocation will be reached after a finite number of rounds. The allocation will be at (or near) one of the endpoints of the “long axis,” that endpoint being the one which is best for the group making the offers. Thus, as long as the individuals on the side of the market receiving offers are not powerless, once the process of making and remaking offers results in the optimal assignment having been found, the process of bargaining and rebargaining within the optimal pairs will begin. Theorem 2 of this paper shows that this process will then converge to an SPB allocation.

It is shown in Rochford [5] that Thompson’s fair division point is an SPB allocation only in special circumstances. For example, in the marriage market discussed in Section 2, it is easily verified that the fair division point is (4, 2, 2, 4) and that this is not an SPB allocation.

Crawford and Knoer’s process is a generalization to the assignment problem formulated as a sidet支付 game of the matching algorithm that Gale and Shapley used to prove the existence of stable assignments when the assignment problem was formulated solely in terms of rankings (cf. p. 1). Crawford and Knoer’s model of the assignment market is formally equivalent to that presented here, except that the characteristic values of the coalitions and all allocations are restricted to integers.
REFERENCES


