Intermittency

Intermittency is, broadly, a phenomenon wherein a given system transitions between two or more qualitatively distinct behaviors over time. It is currently understood under three categories: Pomeau-Manneville dynamics, crisis-induced intermittency, and on-off intermittency. The key distinctions between these three types of intermittency are the kinds of qualitative dynamics observed in each distinct phase as well as the specific dynamics that give rise to these changes.

We begin our analysis with Pomeau-Manneville intermittency, which was introduced in 1980 in a landmark paper by Pomeau and Manneville. The qualitative dynamics of the distinct phases in Pomeau-Manneville intermittency are simple. One phase, the laminar phase, is such that the system appears completely periodic. The second phase is a chaotic burst that arises in a seemingly random way, introducing a chaotic dynamics into the system before the orbit is “re-injected” into the laminar phase, wherein chaos temporarily disappears until the next chaotic burst.

Pomeau-Manneville intermittency is, incidentally, also divided into three distinct categories. Each category, which we will refer as type I, II, and III, relate to the types of bifurcations that give rise to intermittency: the crossing of a saddle node bifurcation, a subcritical Hopf bifurcation, and an inverse period-doubling bifurcation, respectively.

Let us first turn our attention towards type I Pomeau-Manneville intermittency, which is witnessed by both the logistic map and the Lorenz system. Both of these systems exhibit intermittent behavior that results from the variation of a single parameter as this parameter crosses a critical value at which the saddle node bifurcation occurs.
Figures 1 and 2 demonstrate the third iteration of the logistic map crossing the diagonal $y = x$ as the parameter value varies from 3.81 in Figure 1 to 3.84 in Figure 2. We can clearly deduce that for some value $3.81 < a < 3.84$, a saddle node bifurcation occurs when the function crosses the diagonal, at which point previously unstable and stable fixed points coalesce and annihilate. The precise value at which the tangent bifurcation occurs is $1 + \sqrt{8} \approx 3.8284$. In figure 1, the labels $s_1...s_3$ and $u_1...u_3$ indicate the stable and unstable fixed points which exist for the map for values of $a$ greater than or equal to 3.8284. In this window, the period 3 orbit is fully periodic, and does not burst into chaos.

For parameter values just below this critical bifurcation, as in figure 2, intermittency occurs. The space between what were previously points of intersection with the diagonal and the
diagonal itself is the focus of our study of the laminar phase. These are the regions where orbits spend a significant number of iterations in, exhibiting very nearly periodic behavior as a result of their close proximity to the actual period three orbit that exists for very slightly larger parameter values. Of course, as these orbits are not actually periodic, they are eventually mapped out of this laminar region. The number of iterations spent outside of these three regions near the diagonal constitutes the length of the chaotic burst. Eventually, however, the orbit will be re-injected into the laminar region, and the periodic laminar phase begins again. This is the fundamental presentation of Pomeau-Manneville intermittency.

Before turning to a deeper analysis of the laminar phase length, we briefly analyze the action of the bifurcation demonstrated in the Lorenz system, shown in figure 3. This is exactly the same type of Pomeau-Manneville intermittency demonstrated for the logistic map; for a particular value of the parameter $r$, the system experiences a saddle node bifurcation which introduces intermittent chaos, with the “laminar window” again being those iterations for which the orbit is trapped between the curve and the diagonal, near the value at which the bifurcation occurred. This is, of course, a vast oversimplification of the dynamics of the Lorenz system specifically; it only serves as an example of the same exact kind of intermittency arising in a completely different system. Interestingly enough, the variation of other parameters in the Lorenz system can lead to different types of P-M intermittency arising in the same system.

**Laminar Phase Length**

Predicting the precise number of iterations an orbit will spend inside the laminar phase for any given map is unpredictable. The number of iterations that an orbit spends in the laminar phase is dependent on how closely to the vicinity of the actual periodic orbit (as it exists for other parameter values) it is mapped to upon reinjection. Since the value the orbit is mapped to upon reinjection is determined during a chaotic burst, it is not possible to exactly predict at which value the orbit will return to the region. As we shall see, however, a great amount of impressive work has gone into calculating the mean laminar phase for a given map. This mean laminar phase $N$ is accurate with near computer precision, and varies from map to map - we will only
focus on its derivation and mathematical detail for the logistic map specifically. The reasons for
this will become clear in the next few paragraphs.

Pomeau and Manneville themselves first observed that the mean length of the laminar
phases for type I intermittency is proportional to $\varepsilon^{-1/2}$, where $\varepsilon = a - a_t$, the difference between
the current parameter value for a system and the value $a_t$ at which the saddle node bifurcation
occurs. They also conjectured that the laminar phase lengths in type III intermittency are
proportional to $\varepsilon^{1/2}$. As we previously stated, how closely an orbit is mapped to the vicinity of
the periodic orbit determines the length of the laminar phase; but this is dependent on how close
the current parameter value is from the value at which the periodic orbit exists, and while the
former cannot be determined with full precision, this relationship will allow us to rigorously
study the average laminar phase length for a map. We will restrict our attention to a more formal
analysis of the mean laminar phase length for type I intermittency, specifically for the logistic
map, as relatively little work has been done to determine the phase lengths for types II and III.

Since the time of Pomeau and Manneville, a more rigorous approach summarized by
Saramah et. al demonstrates the precise derivation of their observation and further realizes a
constant $k$, relative to the specific map in question, which determines the mean laminar phase
length with precision accuracy.

The derivation is performed for the logistic map specifically, for which the constant $k$ is
similarly determined. We assume that the parameter value used for the parameter is
$1 + \sqrt{8} \approx 3.8284$. The authors first observe that the derivative of the third iterate of the logistic
map for this parameter value is 1 at any fixed point $x_t$. A Taylor expansion is performed for the
third iterate of the logistic map in terms of $x$ about one of these three fixed points.

What follows is that $f^3(x) = x - \varepsilon b_t + \frac{(x-x_t)^2 c_t}{2}$, where $b_t$ is the first partial derivative of
$f^3(x)$ with respect to $a$ and $c_t$ is the second partial derivative of $f^3(x)$ with respect to $x$.

While the values of $b_t$ and $c_t$ differ for different fixed points, the authors show that in
fact their products are always equal. In order to perform some serious calculus with the result
above, the authors use the fact that $x_{n+1} - x_n$ is very small for each $n$ and thus can be
approximated by $dx$. In addition, they note that since the number of iterations spent inside the
laminar channel is very large, we can write $dn = 1$. 

Then the above equation becomes the differential equation \( \frac{dx}{dt} = c_i(x - x_i)^2 + \varepsilon b_i \), from which separation of variables yields the integral \( \int_{x_o}^{x_t} \frac{1}{c_i(x-x_i)^2 + \varepsilon b_i} \, dx = \int_0^N \, dn \). The authors then solve for
\[
N = \frac{1}{\sqrt{\varepsilon/c, b}} \left[ \tan^{-1}( (x_{out} - x_t) \sqrt{\frac{a}{\varepsilon/c, b}} ) - \tan^{-1}( (x_{in} - x_t) \sqrt{\frac{a}{\varepsilon/c, b}} ) \right].
\]
The integral form of this solution is then simplified to
\[
N(y_{in}) = \frac{1}{\sqrt{\varepsilon/c, b}} \left[ \tan^{-1}( y_{out} \sqrt{\frac{a}{\varepsilon/c, b}} ) - \tan^{-1}( y_{in} \sqrt{\frac{a}{\varepsilon/c, b}} ) \right]
\] using the transformation \( y = (x-x_t)/b_t \) from Huberman and Scalapino, a form in terms of the entering and exiting values of the function itself. The new result solves for \( N \) as a function of \( "y_{in}" \), the value at which the orbit enters the channel upon reinjection (or initiation of the orbit).

The ultimate result is that the number of iterations \( N \) that the orbit spends in the laminar phase is approximately given by \( N \equiv \frac{2}{\sqrt{\varepsilon/c, b}} \tan^{-1}( \frac{y_{out}}{N_{in}} ) \). While this is a more powerful, general result, for values of \( y_{out} \) that far exceed \( \sqrt{\frac{\varepsilon}{c}} \), we have the precise equation \( N = \frac{k}{\sqrt{\varepsilon/c, b}} \), where \( k = \frac{\pi}{2} \).

Therefore we can conclude with rigorous confidence that \( N \) is directly proportional to \( \frac{1}{\sqrt{\varepsilon}} \), and that the number of iterations that an orbit spends in the laminar phase is determined by the difference between the parameter value of the map and the critical bifurcation.

**On-Off Intermittency**

On-off intermittent signals have distinct behavior from the other types, which primarily arises from the involvement of a dynamic, rather than fixed parameter. While this distinct characterization of the critical bifurcation parameter is the root cause of the unique presentation of on-off intermittency, Heagy et. al additionally present a geometric mechanism underlying this form of intermittency which provides a much clearer picture of it.

On-off intermittency is studied by Heagy et. al in parametrically driven one-dimensional maps of the form \( y_{n+1} = z_n f(y_n) \) with \( \partial f(y)/\partial y \big|_{0} \neq 0 \) and \( z(\partial f(y)/\partial y) \big|_{y=0} > 1 \), where \( z_n \) is given by some chaotic or random process with the density function \( \rho_z(z) \). The authors present a specific uniformly driven variant of the logistic map in order to characterize on-off intermittency,
given by $f(y_n) = y_n(1 - y_n)$ and $z_n = ax_n$, where $a > 1$ and $x_n$ is a randomly chosen variable in the unit interval.

Whenever $z_n = 1$, this driven variant of the logistic map passes through a transcritical bifurcation at the point $y = 0$, as seen below. Clearly, for $z_n < 1$, the magnitude of the slope of $y_{n+1} = z_ny_n(1 - y_n)$ at $y = 0$ will be less than 1, and so $y = 0$ is stable. For large $z_n$ the magnitude of the slope is greater than 1 and so the point will be unstable.

![Fig. 4](image)

Since the destabilization of this fixed point via the bifurcation seen in figure 4 above only occurs when $z(\partial f(y)/\partial y) |_{y=0} > 1$, this condition is required in order to study intermittency (as otherwise the chaotic bursts themselves will not occur). Fortunately, the partial derivative of $f(y)$ at $y = 0$ can be “absorbed” into the scale of $z_n$ by controlling the parameter $a$ in such a way that, at 0, its value will always be one.

We now discuss the derivation of the critical value of $a$ for which the driven logistic map displays intermittency. Since the intermittent behavior studied depends on the linear instability of the map about $y = 0$, the authors expand $f(y)$ about $y = 0$ to yield $y_{n+1} = z_nf(y_n + O(y_n^2))$. These higher-order nonlinear terms are responsible for the reinjection mechanism that sends the system back to $y = 0$ after a chaotic burst.

While responsible for reinjection, these higher order terms are not necessary for the initiation of a chaotic burst out of a laminar phase, and so we can disregard them in order to greatly simplify the analysis. Disregarding these nonlinear terms of $y_n$ we have $y_n = \prod_{j=0}^{n-1} z_jy_0$. 
The behaviour of \( y \) is determined by the random product of just the \( z_j \) terms from the product above. Since we have a repeated product, it follows that the natural log would serve useful. This leads to defining this random product as a sum.

The condition for the onset of intermittency is \( < \ln z > = 0 \). If this were the case, we would later see \( e^{n<\ln z>} = 1 \). With \( < \ln z > = 0 \) we see that \( y = 0 \) is on average unstable. This causes the intermittent bursts.

Using the law of large numbers we see that this is roughly the average of \( \ln z \) or

\[
\ln P_n = \sum_{j=0}^{n-1} \ln z_j \sim n < \ln z >.
\]

We calculate \( < \ln z > = \int_{z_L}^{z_U} \rho_z(z) \ln z \, dz \), using the phase-space average. We then witness an asymptotic solution in the form of \( y_n \sim e^{n<\ln z>} y_0 \) with \( y_n \) small enough to be directed using a linearized map. Since the logistic map has invariant measure, in this case \( \rho_z(z) = 1 \) and so \( < \ln z > = \ln(a/e) = \ln a - 1 \) which in turn yields \( y_n \sim (a/e)^n y_0 \). We now see that intermittent behaviour occurs when \( a/e = 1 \) or when \( a = a_c = e = 2.71828... \).

For small \( a \), in fact, any \( a < a_c \), it is seen in the figure below that the dynamics of the system remains near zero, while as soon as \( a > a_c \), the system only remains near zero for various laminar phases while intermittently bursting into chaos.

![Fig. 5](image)

Particular attention by the authors is given to the laminar phases of this map; these are witnessed by iterations for which the amplitude of the signal about \( y = 0 \) is very small. A key observation to understanding the dynamics behind the intermittency arising here deeply is that the amplitude of these signals grows smaller as the length of the laminar phase grows longer.
Therefore, if arbitrarily long laminar phases exist, the function will eventually map to zero where it will stay forever. The distribution of laminar phases for on-off intermittency is studied quite carefully by Heagy, beginning with a specific derivation solving exactly for the case of uniform random driving. They show that it is in fact impossible to witness an arbitrarily long laminar phase once intermittent behavior has been initiated.

The precise details of this derivation itself involve heavy probabilistic analysis and are beyond the scope of this report. The authors study the distribution of laminar phases by analyzing the probability that, given that we witness some laminar phase of length at least one iteration, this laminar phase has length exactly \( n \). The key finding is that, at onset, the distribution of laminar phases is given by an asymptotic power law of \(-\frac{\beta}{2}\). This implies that the average laminar phase length is infinite. Beyond onset, however, further manipulation introduces exponential decay into the distribution with respect to \( n \). Thus the distribution of laminar phases for random driven maps decays exponentially with respect to \( n \) as \( n \) approaches infinity. All that this means is that arbitrarily long laminar phases are arbitrarily infrequent. Infinitely long laminar phases do not occur once intermittency has begun - the system always eventually bursts back into chaos again.

Crisis Induced Intermittency

Crisis-induced intermittency is a phenomenon wherein a given system shifts between two or more qualitatively distinct phases, all of which exhibit a chaotic dynamics. This occurs when a system has two or more chaotic attractors which can cross each others’ basin of attraction. This allows for orbits that cross the boundary to spend some number of iterations in one chaotic attractor before crossing over into the next attractor. After some number of iterations, the orbit will then move back into the first attractor, and so on.

It is interesting to note that in crisis induced intermittency, if more than two chaotic attractors are present, it is generally not the case that the phase transitions themselves are periodic. In Pomeau-Manneville intermittency, and in crisis systems with just two attractors, there are exactly two distinct phases which are periodically transitioned between. In the former, we witness phases transition from laminar, to chaotic, to laminar, and so on. In the latter, the
phases shift as the orbit spends time in the first chaotic attractor $C_1$, to the second chaotic attractor $C_2$, then back to $C_1$, and so on.

If more than two chaotic attractors are present, however, the orbit will generally not exhibit a periodic pattern such as $C_1, C_2, C_3, C_1, C_2, C_3, ...$, but rather will transition seemingly randomly between the attractors. This is somewhat reminiscent of a transition diagram for the most simple studies of chaos in one dimensional dynamics. Among $N$ chaotic attractors, $C_i$ to $C_j$ and $C_j$ to $C_i$ are “legal moves” for all distinct values of $i$ and $j$, where $0 \leq i \leq j \leq N$. Therefore we generally witness aperiodic phase transitions, such as $C_1, C_3, C_2, C_3, C_2, C_1, C_3, C_1, C_2, ...$.

A particular case of these aperiodic phase transitions arises in the driven Duffing oscillator, as is shown by Franaszek and Nabaglo; it is noted simply as an example of the above phenomenon occurring beyond a merely theoretical landscape. Crisis induced intermittency is a sort of “odd-duck”, in that, distinct from on-off or Pomeau-Manneville intermittency, there is no laminar phase wherein the system is not chaotic. Rather it merely describes the intermittent transitioning between different sets of chaotic dynamics. It is perhaps purely semantic, but it stimulates the imagination to see how a concept like intermittency can grow, change, and over time branch out in such a way that it loses part of its initial core identity.
Sources:


