Integration – The Substitution Method

Recall the chain rule for derivatives: \[ f(g(x))' = f'(g(x)) \cdot g'(x) \]

Given the composite function \( f(g(x)) \), let's refer to \( g(x) \) as the "inside function".

The "U-Substitution" method of integration is basically the reversal of the chain rule.

In an optimal case, where we could be given an integral of the form \( \int c \cdot f(g(x)) \cdot g'(x) \, dx \), the trick is to recognize that we have an "inside function" \( g(x) \), and the composite function \( f(g(x)) \) is also MULTIPLIED by a nonzero constant multiple (c) of \( g'(x) \), at which point we would factor c out of the integral. (Note that c may not be apparent, but may have to be created by multiplying the integrand by a nonzero constant and its reciprocal.)

This being the case, we can perform the substitution \( u = g(x) \).

It follows that \( \frac{du}{dx} = g'(x) \), and so we'll write \( du = g'(x) \, dx \)

(Using an alternative approach, we could solve the equation \( du = g'(x) \, dx \) for \( dx \) and substitute.)

Either way, \( \int c \cdot f(g(x)) \cdot g'(x) \, dx = c \int f(u) \cdot du \)

Examples

Simplest (we have the inside function’s exact derivative)

1) \( \int 2x(x^2 - 7)^9 \, dx \)  
2) \( \int \cot x \, dx \)  
3) \( \int \frac{1}{x\sqrt{5 + \ln x}} \, dx \)  
4) \( \int \sin^5 x \cdot \cos x \, dx \)

A bit tougher (we have a constant multiple of the inside function’s derivative)

5) \( \int_1^2 (2x + 1)^5 \, dx \)  
6) \( \int e^x - e^{-x} \, dx \)  
7) \( \int_{-2}^2 x(x^2 + 4)^3 \, dx \)

8) \( \int 3e^{2x}\sqrt[3]{4 - 8e^{2x}} \, dx \)

More steps involved (we perform the original substitution, but there are still X’s left)

Here we must use our original substitution \( u = g(x) \) to solve for \( x \) in terms of \( u \).

10) \( \int_0^5 x\sqrt{x + 4} \, dx \)  
11) \( \int \frac{x}{\sqrt{1 - x}} \, dx \)  
12) \( \int \frac{x}{1 + x^4} \, dx \)  
13) \( \int \frac{x^5}{(x^2 + 1)^5} \, dx \)

Solutions to examples

1) \( \frac{1}{10} (x^2 - 7)^{10} + c \)  
2) \( \ln |\sin x| + c \)  
3) \( 2\sqrt{5 + \ln x} + c \)  
4) \( \frac{1}{6} \sin^6 x + c \)  
5) \( \frac{3724}{3} \)  
6) \( \ln(e^x + e^{-x}) + c \)
7) \( 0 \) (Note: Odd Function)  
8) \(-\frac{9}{64} \left(4 - 8e^{2x}\right)^4 + c\)  
9) \(\frac{1}{12} \sec^4(3x) + c\)  
10) \(\frac{506}{15}\)  
11) \(-2\sqrt{1-x} + \frac{2}{3} \sqrt{1-x}^3 + c\)  
12) \(\frac{1}{2} \tan^{-1}(x^2) + c\)  
13) \(-\frac{1}{4(x^2 + 1)^2} + \frac{1}{3(x^2 + 1)^3} - \frac{1}{8(x^2 + 1)^3} + c\)