Wave Equation Problems

1. (a) If $F(s)$ is a $C^2$-function of the variable $s$, find a condition on the vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ so that

$$u(x_1, x_2, x_3, t) = F(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 - t)$$

is a solution to $u_{tt} = c^2 \nabla^2 u$ in $\mathbb{R}^3 \times \mathbb{R}$. (Such solutions are called plane waves and are constant on the planes $\alpha \cdot x - t = \text{constant}$.)

(b) Find the relationship which must hold between the initial data $f(x)$ and $g(x)$ for a plane-wave solution.

(c) Find all plane wave solutions with initial condition $u(x, y, z, 0) = x - y + 1$.

2. Find the solution of the initial value problem

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}$$
$$u(x, y, z, 0) = x^2 + y^2 \quad u_t(x, y, z, 0) = 0$$

using Both the three-dimensional and two-dimensional Kirchhoff formulas.

3. Find a formula for the solution $v(x, y, t)$ of the Cauchy problem for the two-dimensional Klein-Gordon equation:

$$v_{tt} = c^2(v_{xx} + v_{yy}) - m^2 v \quad t > 0, x, y \in \mathbb{R}^2$$
$$v(x, y, 0) = f(x, y) \quad u_t(x, y, 0) = g(x, y).$$

Hint: Find the PDE $u(x, y, z, t) = \cos\left(\frac{m}{c} z\right) v(x, y, t)$ solves.

4. (a) For $n = 3$, suppose that $f, g \in C_0^\infty(\mathbb{R}^3)$ and solve the three-dimensional wave equation in $\mathbb{R}^3$ with the solution discussed in class. Show that $|u(x, t)| \leq C/t$ for some constant $C$ and for all $x \in \mathbb{R}^3$ and $t > 0$.

(b) Is a similar result true for $n = 2$?

5. Consider the transport equation in $\mathbb{R} \times \mathbb{R}^+$

$$u_t + u_x = 0 \quad u(x, 0) = f(x).$$

Develop a complete theory for the PDE - define a weak solution (carefully including the initial data), give examples of solutions which are weak but not classical solutions. Either prove the solutions are unique or give counter examples. Provide a regularity theorem.

6. Consider the invicid Burgers equation in $\mathbb{R}$ and $t > 0$

$$u_t + uu_x = 0 \quad u(x, 0) = f(x).$$

Define a weak solution for this PDE. Show that

$$u_1 = \begin{cases} 0, & x/t < \frac{1}{2} \\ 1, & x/t > \frac{1}{2} \end{cases} \quad u_2 = \begin{cases} 0, & x < 0 \\ x/t, & 0 < x/t < 1 \\ 1, & x/t > 1 \end{cases}$$

are both weak solutions to Burgers equation. Notice $u_i(x, 0) = 0$ for $x < 0$ and $u_i(x, 0) = 1$ for $x > 0$ for both $i = 1, 2$ - hence weak solutions are not unique.