1. Formally show the equation $u_t = \nabla^2 u$, $u(x, 0) = g(x)$ on $\mathbb{R}^N$ may not have a unique solution. That is, formally check that

$$u(x, t) = \sum_{k=0}^{\infty} \frac{1}{2k!} x^{2k} \frac{d^k}{dt^k} e^{-1/t^2}$$

satisfies $u_t = u_{xx}$, $u(x, 0) = 0$ for $t > 0$, $x \in \mathbb{R}$. (Apparently proving the series converges is not easy; see F John, PDEs, Springer-Verlag.)

2. Suppose $g(x)$ is bounded and continuous on $\mathbb{R}^N$. We know

$$u(x, t) = \frac{1}{(4\pi k^2)^{n/2}} \int_{\mathbb{R}^N} e^{-|x-y|^2/4k} g(y) dy$$

is a $C^\infty(\mathbb{R}^N \times (0, \infty))$ solution to $u_t = \nabla^2 u$, $u(x, 0) = g(x)$. Show

(a) $|u(x, t)| \leq \sup \{|g(y)| : y \in \mathbb{R}^N\}$

(b) If in addition, $\int_{\mathbb{R}^N} |g(y)| dy < \infty$, show $\lim_{t \to \infty} u(x, t) = 0$ uniformly in $\mathbb{R}^N$.

3. (Related to Prob 13, P.88 in your book) Heat conduction is a semi-infinite rod with initial temperature $g(x)$: Solve $u_t = u_{xx}$ for $x, t > 0$ and $u(x, 0) = g(x)$ subject to the following boundary conditions:

(a) Assume $g(0)=0$ and the rod is maintained at zero temperature at $x = 0$. That is, $u(0, t) = 0$ for $t > 0$.

(b) Suppose the rod is insulated at $x = 0$. That is, $u_x(0, t) = 0$ for $t > 0$. Find a formula for $u$. Does $g'(0) = 0$ need to be required?

4. (Essentially Problem 12 P. 87) Consider the PDE

$$u_t = \nabla^2 u - u \quad \text{for} \quad t > 0, \ x \in \mathbb{R}^N$$

$$u(x, 0) = g(x) \quad \text{for} \quad x \in \mathbb{R}^N.$$  

Find a formula for $u$ if $g$ is continuous and bounded. Is the solution bounded? Is it unique? (Hint: consider $v = e^u$.)

5. Apply the similarity method to the linear transport equation $u_t + au_x = 0$ to obtain the special solutions $u(x, t) = c(x - at)^\alpha$.

6. Consider the one-dimensional heat equation with Robin Boundary conditions:

$$u_t = u_{xx} + f(x) \quad 0 < x < 1, \ t > 0$$

$$u(0, t) - \kappa u_x(0, t) = 0, \quad t > 0$$

$$u(1, t) + \kappa u_x(1, t) = 0, \quad t > 0.$$  

By home-cooking your own version of the Poincaré Inequality, prove the solution, if it exists (using MAT 476 methods you can show it does), converges exponentially in time to its steady state.