1. Let $\mathcal{H}$ be the vector space consisting of $2\pi$ periodic functions on $\mathbb{R}$. Put the $L^2$ topology on this vector space. Then define the Fourier coefficient of a function in $\mathcal{H}$ to be $c_n = (u, e^{inx})$ with $(\cdot, \cdot)$ is the inner product on $L^2(\mathbb{T})$. Prove

$$u \in H^{-1}(\mathbb{T}) \iff \sum_{k \in \mathbb{Z}} |k|^{-2} |c_k|^2 < \infty.$$ 

2. Prove the Sobolev imbedding theorem - $H^1_0(\mathbb{T}) \subset C^{0,1/2}(\mathbb{T})$ (space of Hölder continuous functions with $\theta < 1/2$). Specifically show

$$|u(x) - u(y)| \leq C_\theta |x - y|^\theta \|u\|_{H^1(\mathbb{T})}.$$ 

Find an explicit upper bound for $C_\theta$. Hint: Start by showing $|e^{ia} - e^{ib}| \leq 2 |a - b|^\theta$.

3. Consider the Chaffee-Infante Equation

$$\partial_t u - u_{xx} + u^3 - u = f, \quad 0 < x < L, \ t > 0$$

$$u(0, t) = u(L, t) = 0, \ t > 0$$

$$u(x, 0) = g(x).$$

Formulate a definition for a (the?) weak solution and prove the weak solution exists. Prove uniqueness of your weak solution.

4. (Corollary not proven correctly in class) Suppose $\{u_m\}$ converges weakly in $L^2(0, T; H^1_0(0, L))$ to $u$. That is,

$$\int_0^T (u(t), v)_{H^1} dt \to \int_0^T (u(t), v)_{H^1} dt$$

(1)

as $m \to \infty$ and for all $v \in L^2(0, T; H^1_0(0, L))$. Also assume $H^1_0(0, L)$ is compactly imbedded in $L^2(0, L)$.

(a) Show that (1) implies a subsequence, $\{u_{m'}\}$, converges weakly to $u$ in $H^1_0(0, L)$ almost everywhere on $[0, T]$, and hence $u_{m'} \to u$ in $L^2(0, L)$ almost everywhere on $[0, T]$.

(b) Show $\int_0^T \|u_{m'} - u\|_{L^2}^2 dt \to 0$ as $m' \to \infty$. 