Green's Function Problems (Elliptic PDEs)

Problems starred are required.

1*. Suppose

\[ \Phi(x, y) = \begin{cases} \frac{-1}{2\pi} \ln |x - y| & n = 2, \\ \frac{1}{(n-2)4\pi|x-y|^{n-2}} & n \geq 3. \end{cases} \]

If \( f \in C^k(\mathbb{R}^n) \) with compact support, show that \( u(x) = \int_{\mathbb{R}^n} \Phi(x, y)f(y)d\Omega y \in C^{k+1}(\mathbb{R}^n) \).

2*. Let \( \Omega = \{(x, y) \in \mathbb{R}^2 : 0 < y, 0 < x \} \). Solve, by constructing a Green's function,

\[ \nabla^2 u = 0 \quad 0 < x, \quad 0 < y, \]

\[ u(x, 0) = f(x), \quad 0 < x, \quad u(0, y) = 0, \quad 0 < y. \]

State and prove a regularity theorem for \( u \). What’s require of \( f \) for the solution to be smooth? Is the solution unique? Prove or disprove.

3. Let \( \Omega \) be the semi-circle \( (0 < r < a, \ 0 < \theta < \pi) \). Find the Green's function \( G(P, Q) \) satisfying

\[ \nabla^2 G = \delta(|P - Q|), \]

with \( G(P, Q) = 0 \) for \( P \) on the boundary.

4. Given the delta function \( \delta(x)\delta(y)\delta(z - z_0) \), find the equivalent delta function in spherical coordinates.

5. Solve the following ODE by constructing a Green’s function

\[ y'' + y = f(x) \]

\[ y(0) = 0 = y'(1). \]

How does the regularity of the solution depend on \( f \)? A proof is required.