First-Order PDEs

We study equations of the form

\[(2.1) \quad a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y),\]

\[(2.2) \quad a(x, y)u_x + b(x, y)u_y = c(x, y, u),\]

\[(2.3) \quad a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u).\]

Equations (2.1), (2.2), and (2.3) are called linear, semi-linear, and quasi-linear first order partial differential equations respectively. While we have written them in two space dimensions, the theory outlined works in any dimension. We suppose the solution is prescribed on a curve, \(\Gamma\), in \(\mathbb{R}^2\). This together with the PDE is called a Cauchy Problem.

As with any PDE we have four major concerns we like to address:

1. Does a solution exist?
2. Is the solution unique?
3. Does solution depend continuously on the data?
4. What are the properties of the solutions?

If the answer to the first three is yes, we say the PDE is \textit{well-posed}.

2.1. Motivation

To see a source of first-order PDEs consider traffic flow on the \(x\) axis. Let \(\rho(x, t)\) be the density of cars (car/length). Then the number of cars in the interval \(I = [x_0, x]\) at time \(t\) is

\[
\text{number of cars in I} = \int_{x_0}^{x} \rho(s, t) \, ds.
\]

We set \(q(x, t)\) to be the flux of cars; that is, the number of cars per unit time passing \(x\) at time \(t\) with \(q\) positive for cars moving in increasing \(x\). If we assume cars do not evaporate, or turn off the street, or multiply, then

\[
\frac{d}{dt} \int_{x_0}^{x} \rho(s, t) \, ds = q(x_0, t) - q(x, t).
\]
This last statement is an example of a conservation law and it is quite general. If we assume the time derivative and integral commute, (we will investigate in great detail later), we get
\[ \int_{x_0}^{x} \rho_t(s,t) \, ds = q(x_0,t) - q(x,t), \]
where \( \rho_t \) is shorthand for \( \frac{\partial \rho}{\partial t} \). If \( \rho_t \) is continuous we may apply the fundamental theorem of calculus to conclude
\[ \rho_t(x,t) + q_x(x,t) = 0. \]
Equation (2.4) represents the local form of a conservation law. Its form is quite common in physics and engineering.

To make the situation more applicable to cars, suppose \( v(x,t) \) is the velocity of cars at position \( x \) and time \( t \). We assume that the velocity is a function of the density. That is, \( v(x,t) = F(\rho) \) for some function \( F \). Thus the flux \( q(x,t) = v(x,t)\rho(x,t) \) is also a function of the density. We will model human behavior in two ways.

- (The LA assumption) Suppose \( v(x,t) = c_0 \) a constant and independent of density. Thus cars do not slow down regardless of the density of traffic around them. Then \( q = c_0\rho \), and (2.4) becomes
  \[ \rho_t + c_0\rho_x = 0 \]
  \[ p(0,x) = \rho_0(x). \]
  Here \( \rho_0 \) is the initial traffic. We will show a method to solve this in a moment. For now we can guess (think a bit) solution to be
  \[ \rho(x,t) = \rho_0(x - c_0t). \]
Check that this solves the PDE.

- (A more reasonable assumption) This time we suppose, more realistically of human behavior,
  \[ v(x,t) = c_0 \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right), \]
where \( c_0 \) and \( \rho_{\text{max}} \) are known constants. Since \( q = v\rho \), we find
  \[ q_x(x,t) = c_0\rho_x \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) - c_0 \frac{\rho \rho_x}{\rho_{\text{max}}}. \]
  Thus our conservation law becomes
  \[ \rho_t - \frac{2c_0}{\rho_{\text{max}}} \rho \rho_x + c_0\rho_x = 0. \]
If we set \( u = 1 - 2\rho/\rho_{\text{max}} \) and rescale the time by setting \( y = c_0t \), the PDE becomes \( u_y + uu_x = 0 \) - which is a well-studied PDE called Burgers’ equation. We will solve it shortly. This model can predict the wave motion (stop and go traffic) we experience on the highway.
2.2. Method of Characteristics

To be more definite, we consider the semi-linear, first-order, Cauchy problem

\[ a(x, y)u_x + b(x, y)u_y = c(x, y, u) \]

\[ u|_\Gamma = u_0, \]

where \(a, b, c, \Gamma,\) and \(u_0\) are all given.

To solve the Cauchy problem, (2.5), note that \(u_x\) and \(u_y\) appear linearly in the PDE. Thus the PDE resembles the chain rule. Recall

**Theorem 2.1.** Suppose \(u(x, y) : \mathbb{R}^2 \to \mathbb{R}\) and that both \(u_x\) and \(u_y\) exist on \(\mathbb{R}^2\). If \(f(\tau) = u(x(\tau), y(\tau))\) with \(x(\tau), y(\tau)\) differentiable, then

\[ \frac{df}{d\tau} = \frac{\partial u}{\partial x} \frac{dx}{d\tau} + \frac{\partial u}{\partial y} \frac{dy}{d\tau} \]

\[ = \frac{dx}{d\tau} u_x + \frac{dy}{d\tau} u_y. \]

**Proof.** See Theorem 1.29 in Chapter One. \(\square\)

If we compare this with above PDEs it suggests the following procedure: at each point on the curve \(\Gamma\), where the solution is known, we integrate off \(\Gamma\) by solving

\[ \frac{dx}{d\tau} = a(x, y), \quad \frac{dy}{d\tau} = b(x, y) \]

and with both \(x(\tau)\) and \(y(\tau)\) initially on \(\Gamma\). Then, by the chain rule,

\[ \frac{du}{d\tau} = \frac{dx}{d\tau} u_x + \frac{dy}{d\tau} u_y \]

\[ = a(x, y)u_x + b(x, y)u_y \]

\[ = c(x, y, u). \]

That is, at each point on \(\Gamma\), we turn the PDE into an ODE!

We make this procedure into an algorithm which we will justify later.

(1) Parameterize the curve \(\Gamma\) as \((x_0(s), y_0(s))\) for \(s\) in some interval.

(2) For each \(s\) solve the system of ODEs given by

\[ \frac{dx}{d\tau} = a(x, y), \quad \frac{dy}{d\tau} = b(x, y) \]

\[ x(0) = x_0(s) \quad y(0) = y_0(s). \]

(3) The PDE is now an ODE along \((x(\tau), y(\tau))\) given by

\[ \frac{du}{d\tau} = c(x, y, u) \]

\[ u(0) = u_0(x_0(s), y_0(s)). \]

The curves generated by \(x(\tau), y(\tau)\) are called **Characteristic Curves**.

(4) Solve the ODE in (3). The solution will now be expressed in terms of \(\tau\) and \(s\). Invert the solutions found in (2) to solve for \(\tau = \tau(x, y)\) and \(s = s(x, y)\).

Insert this in (3) to express \(u = u(x, y)\), the solution to the Cauchy problem.

The liner and quasi-linear PDEs work in a similar way.
Example 2.2. Solve the Cauchy problem

\[ u_x + u_y = 0 \]
\[ u(x, 0) = f(x), \]

where \( f \) is some smooth function on \( \mathbb{R} \).

Solution. The curve \( \Gamma \) here is the \( x \) axis. Thus we set \( x_0(s) = s \) and \( y_0(s) = 0 \).

Following the algorithm above, we solve

\[ \frac{dx}{d\tau} = 1 \]
\[ \frac{dy}{d\tau} = 1 \]
\[ x(0) = s \]
\[ y(0) = 0. \]

The solution is \( x(\tau) = \tau + s \) and \( y(\tau) = \tau \). The characteristic curves are thus given by \( y = x - s \). Step (3) gives the ODE

\[ \frac{du}{d\tau} = 0 \]
\[ u(0) = f(s). \]

Its solution is \( u(\tau, s) = f(s) \). Next we invert \( x = \tau + s \) and \( y = \tau \) to find \( \tau = y, s = x - y. \) Thus the solution is \( u(x, y) = f(x - y) \). Check this is the solution!

Example 2.3. Solve the linear Cauchy problem

\[ u_x + u_y + u = 1 \]
\[ u = \sin x, \]
on \( y = x + x^2 \), with \( x > 0 \).

Solution. The curve \( \Gamma \) may be parameterized by letting \( x = s \) and \( y = s + s^2 \) with \( s > 0 \). We need to solve

\[ \frac{dx}{d\tau} = 1 \]
\[ \frac{dy}{d\tau} = 1 \]
\[ x(0) = s \]
\[ y(0) = s + s^2. \]

The solution is \( x(\tau) = \tau + s \) and \( y(\tau) = \tau + s + s^2 \). Step (3) gives the ODE

\[ \frac{du}{d\tau} = 1 - u \]
\[ u(0) = \sin s. \]

We may rewrite this as \( \frac{du}{d\tau} + u = (e^\tau u)' = 1 \cdot e^\tau \). Integrating, we find

\[ u(\tau, s) = (e^\tau + c)e^{-\tau} \]
\[ = 1 + (\sin s - 1)e^{-\tau}. \]

Finally, eliminating \( \tau \) between \( x \) and \( y \), we find \( y = x + s^2 \). Thus the characteristics are 45 degree lines. Solving for \( s \), we find \( s = \sqrt{y - x} \) (positive root since \( s > 0 \)). It follows \( \tau = x - \sqrt{y - x} \), and the solution is

\[ u(x, y) = 1 + (\sin \sqrt{y - x} - 1)e^{-\sqrt{y - x}}. \]
2.3. Homework

Exercise 2.1. Consider the traffic flow problem with initial data corresponding to traffic density that results after an infinite line of stopped traffic is started by a red light turning green. Suppose \( v(\rho) = 1 \) (LA assumption). Consider the Cauchy problem

\[
\rho_t + \rho_x = 0, \\
\rho(x, 0) = \begin{cases} 
1 & x < -1 \\
-x & -1 < x < 0 \\
0 & x > 0.
\end{cases}
\]

(a) Sketch several typical characteristics.
(b) Find the solution.
(c) Where is the solution defined?
(d) Sketch the density at \( t = 0, 1, 2 \).

Exercise 2.2. Repeat Problem 1 with \( v(\rho) = (1 - \rho) \). That is, solve

\[
\rho_t - 2\rho \rho_x + \rho_x = 0
\]

with the above initial data.
(a) Sketch several typical characteristics.
(b) Find the solution.
(c) Where is the solution defined?
(d) Sketch the density at \( t = 0, 1, 2 \). Is the result consistent with your experiences with traffic in this situation?

2.4. More Examples

Here are some more well behaved examples.

Example 2.3. Consider the Cauchy problem for \( x > 0 \)

\[
u_t + u_x = 0
\]

with boundary condition

\[
u(0, t) = g(t)
\]

and with initial data

\[
u(x, 0) = f(x).
\]

Note this is the same as taking \( \Gamma \) to be the positive \( t \) and positive \( x \) axis. Find necessary conditions on the data so that the solution is at least continuously differentiable.

Solution. In this problem there is a discontinuity in the smoothness of the curve \( \Gamma \). We break \( \Gamma \) into two parts \( \Gamma = \Gamma_1 \cup \Gamma_2 \) with \( \Gamma_1 \) the \( x \) axis and \( \Gamma_2 \) the \( t \) axis.
• On $\Gamma_1$: we set $x = s$ and $t = 0$ with $s > 0$. The characteristics solve
\[
\frac{dx}{d\tau} = 1; \quad x(0) = s
\]
\[
\frac{dt}{d\tau} = 1; \quad t(0) = 0.
\]
The solution is $x(\tau) = \tau + s$, and $t(\tau) = \tau$. Along the characteristics the PDE becomes the ODE
\[
\frac{du}{d\tau} = 0; \quad u(0) = f(s).
\]
Hence $u(\tau, s) = f(s)$. Now we have to invert. We see $\tau = t$ and $s = x - t$. Thus, for $s > 0$ or $x > t$ the solution is $u(x, t) = f(x - t)$.

• On $\Gamma_2$: we set $x = 0$ and $t = s$ with $s > 0$. The characteristics solve
\[
\frac{dx}{d\tau} = 1; \quad x(0) = 0
\]
\[
\frac{dt}{d\tau} = 1; \quad t(0) = s.
\]
The solution is $x(\tau) = \tau$, and $t(\tau) = \tau + s$. Along the characteristics the PDE becomes the ODE
\[
\frac{du}{d\tau} = 0; \quad u(0) = g(s).
\]
Hence $u(\tau, s) = g(s)$. Now we have to invert. We see $\tau = x$ and $s = t - x$. Thus, for $s > 0$ or $t > x$ the solution is $u(x, t) = g(t - x)$.

We have
\[
(2.6) \quad u(x, t) = \begin{cases} 
  f(x - t) & x > t \\
  g(t - x) & t < x
\end{cases}
\]
The discontinuity in the smoothness of $\Gamma$ at the origin requires some compatibility conditions on the data for the solution to be differentiable. If we want the solution to be continuous at $x = t$, we need to require $f(0) = g(0)$. For the solution to be differentiable we need $\partial u/\partial x$ to match at $x = t$. This requires $f'(0) = -g'(0)$. Requiring $\partial u/\partial t$ to match at $x = t$ gives the same condition. Summarizing, if $f$ and $g$ are $C^1(\mathbb{R}^+)$, $f(0) = g(0)$, and $f'(0) = -g'(0)$, then the (unique?) solution is given by (2.6).

There is no reason to stick to two dimensions.

**Example 2.4.** Consider the Cauchy problem
\[
xu_x + yu_y + u_z = u
\]
\[
u(x, x, 0) = h(x, y).
\]

**Solution.** In this situation $\Gamma$ is a surface. We parameterize it by setting $x = s_1$, $y = s_2$ and $z = 0$ with $s_1, s_2 \in \mathbb{R}$. The characteristics solve
\[
\frac{dx}{d\tau} = x, \quad x(0) = s_1,
\]
\[
\frac{dy}{d\tau} = y, \quad y(0) = s_2,
\]
\[ \frac{dz}{d\tau} = 1, \quad z(0) = 0, \]

and along the characteristic surfaces the PDE is the ODE

\[ \frac{du}{d\tau} = u, \quad u(0) = h(s_1, s_2). \]

Solving, we find \( x(\tau) = s_1 e^{\tau}, y(\tau) = s_2 e^{\tau}, z(\tau) = \tau, \) and \( u(\tau) = e^{\tau} h(s_1, s_2). \)

Now we have to invert. We find \( z = \tau, \quad s_1 = x e^{-z}, \) and \( s_2 = y e^{-z}. \) Thus

\[ u(x, y, z) = e^{z} h(x e^{-z}, y e^{-z}). \]

**Example 2.5.** Solve the Cauchy problem

\[ -2xyu_x + 4xu_y + yu = 4xy \]

with \( u = 2\sqrt{x} \) on \( x > 0, \ y = 0. \)

**Solution.** We parameterize \( \Gamma \) by \( x = s \) and \( y = 0 \) with \( s > 0. \) Since \( x > 0, \) we first divide the PDE by \( x \) to simplify it. Then, we need to solve

\[ \frac{dx}{d\tau} = -2y, \quad x(0) = s, \]

\[ \frac{dy}{d\tau} = 4, \quad y(0) = 0, \]

and along the characteristic surfaces the PDE is the ODE

\[ \frac{du}{d\tau} = 4y - \frac{yu}{x}, \quad u(0) = 2\sqrt{x}. \]

Solving the \( y \) equation we get \( y(\tau) = 4\tau. \) Then the equation for \( x \) can be solved and we see \( x(\tau) = -4\tau^2 + s. \) To find the characteristic curves we eliminate \( \tau \) by solving for \( \tau. \) We see \( \tau = y/4 \) and so \( x = -\frac{y^2}{4} + s. \) This shows the characteristic curves backward facing parabolas with apex at \( x = s. \)

To find the solution we put the values we found for \( x \) and \( y \) into the equation for \( u. \) We find

\[ \frac{du}{d\tau} + \frac{4\tau}{-4\tau^2 + s} u = 16\tau \]

The integrating factor is

\[ \mu = \exp\left(-\frac{1}{2} \ln(-4\tau^2 + s)\right) \]

\[ = (-4\tau^2 + s)^{-1/2}. \]

Thus

\[ \frac{d((-4\tau^2 + s)^{-1/2}u)}{d\tau} = 16\tau(-4\tau^2 + s)^{-1/2}. \]

Integrating we find

\[ u(\tau, s) = \left(-4(-4\tau^2 + s)^{1/2} + c(s)\right)(-4\tau^2 + s)^{1/2}. \]

Here \( u(0, s) = 2\sqrt{s}. \) Thus

\[ u(\tau, s) = -4(-4\tau^2 + s) + (2 + 4\sqrt{s})(-4\tau^2 + s)^{1/2}. \]
Finally, inverting we get
\[ u(x, y) = -4x + \left(2 + 4\sqrt{x + \frac{y^2}{4}}\right)\sqrt{x}. \]

2.5. Justification of the Method of Characteristics

In this section justify our algorithm for solving the Cauchy problem. We again limit the arguments to the two-dimensional case. The extension to higher dimensions is straightforward. Moreover, we also only consider the quasi-linear PDE as both the linear and semi-linear equations are special cases. To solve for the solution and characteristics we need to know system of ODEs

\[
\begin{align*}
\frac{dx}{d\tau} &= a(x, y, u), \\
\frac{dy}{d\tau} &= b(x, y, u), \\
\frac{du}{d\tau} &= c(x, y, u),
\end{align*}
\]

\[ x(0) = x_0(s), \quad y(0) = y_0(s), \quad u(0) = u_0(s). \]

has a solution. To do this we apply the fundamental theorem of ODEs to our situation. To set up the theorem, suppose \( \Gamma = \{(x_0(s), y_0(s)) | s \in I\} \) for some closed interval \( I \), and set, for a given \( d > 0 \) and each \( s \in I \),

\[ \Omega_s := \{(x, y, u) | (x - x_0(s))^2 + (y - y_0(s))^2 + (u - u_0(s))^2 < d^2 \}. \]

Set \( \Omega = \bigcup_{s \in I} \Omega_s \). Let \( M \) bound \( a, b, \) and \( c, \) and any of its derivatives on \( \Omega \).

**Theorem 2.6.** Suppose \( a(x, y, u), b(x, y, u), c(x, y, u) \) are all continuously differentiable (each partial derivative is continuous) on \( \Omega \) for some \( d > 0 \). Then a positive constant, \( C \), which depends only on \( d \) and \( M \), exists so that the system of ODEs, (2.7) has a unique solution for (at least) \(-C \leq \tau \leq C\).

**Proof.** For a proof of this theorem (at a MAT 371 level) see the appendix in Hirsch, Smale, and Devaney\(^1\). We note that, the constant \( C \) in the theorem increases as \( d \) increases or as \( M \) decreases.

To give the idea of the proof suppose we wanted to show \( x'(t) = f(x(t)), x(0) = x_0 \) has a solution. We first turn the ODE into an integral equation. By integrating the following two problems are equivalent:

\[
\begin{align*}
\frac{dx}{dt} &= f(x), \quad x(t) = x_0 + \int_0^t f(x(s)) \, ds \\
x(0) &= x_0
\end{align*}
\]

The integral form of the problem has the advantage that we need only show the solution is continuous. Then Fundamental Theorem of Calculus shows the solution is differentiable. To show the integral equation has a solution we employ a Picard iteration: Suppose \( \phi_0 \) is given (it could be zero for example). We set

\[ \phi_n(t) = x_0 + \int_0^t f(\phi_{n-1}(s)) \, ds. \]

By shrinking the time the sequence, \{\phi_n\}, generated can be shown to be Cauchy. Such sequences are known to converge. The limit, \phi, can be shown to solve the integral equation, and hence, the differential equation.

Next we justify the inversion of process of the algorithm. We state without proof the Inverse Function Theorem in the context we need it. If we are successful in solving (2.7), we obtain \(x = x(\tau, s)\) and \(y = y(\tau, s)\). We need to invert to solve for \(\tau = \tau(x, y)\) and \(s = s(x, y)\) near \(\Gamma\). We need conditions only on the known data \((a, b, c, \Gamma)\) to ensure the inverse exists. We cannot have conditions on solutions we may not explicitly be able to find. If we linearize the system at a point on \(\Gamma\), say \(\tau = 0\) and \(s = s_0\), we find

\[
\begin{align*}
x(\tau, s) &\approx x(0, s_0) + \frac{\partial x}{\partial \tau}(0, s_0)(\tau - 0) + \frac{\partial x}{\partial s}(0, s_0)(s - s_0) \\
y(\tau, s) &\approx y(0, s_0) + \frac{\partial y}{\partial \tau}(0, s_0)(\tau - 0) + \frac{\partial y}{\partial s}(0, s_0)(s - s_0).
\end{align*}
\]

We may write this as

\[
\begin{pmatrix}
x(\tau, s) \\
y(\tau, s)
\end{pmatrix} \approx \begin{pmatrix}
x(0, s_0) \\
y(0, s_0)
\end{pmatrix} + \begin{pmatrix}
x_{\tau} & x_s \\
y_{\tau} & y_s
\end{pmatrix} \begin{pmatrix} \tau \\ s - s_0 \end{pmatrix}.
\]

We see the linear system can be inverted if and only if the matrix in the above expression is invertible. We might hope the invertibility of the nonlinear system inherits the invertibility of the linearized system. At least in a region where the linear system is a good approximation to original system. This is indeed the case.

**Theorem 2.7.** (Inverse Function Theorem) Suppose \(x(\tau, s)\) and \(y(\tau, s)\) are continuously differentiable near \(\tau = 0\) and \(s = s_0\) and that the determinant of the Jacobian is non zero. That is, at \(\tau = 0\), \(s = s_0\),

\[
\begin{vmatrix}
x_{\tau} & x_s \\
y_{\tau} & y_s
\end{vmatrix} \neq 0.
\]

Then there exists a neighborhood of \(\tau = 0\), \(s = s_0\) such that \(x(\tau, s), y(\tau, s)\) is invertible. Moreover, the inverse is continuously differentiable.

For the definition of a neighborhood, see Definition 1.9. For a proof of this theorem see Bartle\(^2\). We emphasis that Theorem 2.7 is local. Even if the Jacobian in non vanishing everywhere, it does NOT imply the equations are invertible everywhere.

**Example 2.8.** Consider

\[
x = e^\tau \cos(s), \quad y = e^\tau \sin(s).
\]

The Jacobian is \(e^\tau \neq 0\). However, \((\tau = 0, s = 0)\) and \((\tau = 0, s = 2\pi)\) get mapped to the same point \((x = 1, y = 0)\). So the map is not injective, and hence not invertible. The Inverse-Function Theorem is local.

Thus a sufficient condition for the inverse to exist, at least locally, is that the Jacobian does not vanish along \( \Gamma \). Fortunately we can express the Jacobian in terms of our known quantities. Indeed, in the semilinear case,

\[
\begin{vmatrix}
  x_\tau & x_s \\
  y_\tau & y_s \\
\end{vmatrix}_{\tau=0} = \begin{vmatrix}
  a(x_0(s), y_0(s)) & x'_0(s) \\
  b(x_0(s), y_0(s)) & y'_0(s) \\
\end{vmatrix}.
\]

To apply the inverse function theorem we need to know that \( x(\tau, s) \) and \( y(\tau, s) \) are continuously differentiable with respect to \( \tau \) and \( s \) near \( \Gamma \). The Fundamental theorem of ODEs gives the differentiability with respect to \( \tau \). We need to investigate differentiability with respect to \( s \).

**Example 2.9.** Suppose we wanted to solve the ODE

\[
\frac{dy}{d\tau} + y = 1, \\
y(0) = y_0(s).
\]

That is, we solve an ODE in which the initial data depends on a parameter \( s \). What can we say about \( \partial y / \partial s \)? In this case we can solve for \( y \) and find out. Here

\[
y(\tau, s) = 1 + (y_0(s) - 1)e^{-\tau}.
\]

As long as the given data \( y_0 \) is differentiable, we find

\[
\frac{\partial y}{\partial s} = y'_0(s)e^{-\tau}.
\] (2.8)

In general we cannot solve for \( y \) to check if it is differentiable with respect to \( s \). We need to find another way to show it is. Suppose we simply took the derivative of the original ODE with respect to \( s \). We find

\[
\frac{d}{d\tau} \left( \frac{\partial y}{\partial s} \right) + \frac{\partial y}{\partial s} = 0, \\
y(0) = y'_0(s).
\]

To make this more attractive let’s call \( \eta(\tau) = \frac{\partial y}{\partial s} \). Then \( \eta \) solves

\[
\frac{d\eta}{d\tau} + \eta = 0, \\
\eta(0) = y'_0(s).
\]

The solution is \( \eta(\tau) = y'_0(s)e^{-\tau} \), in agreement with (2.8). Thus to show \( y \) is differentiable with respect to \( s \), we need the solution to the equation for \( \eta \) to exist. We can summarize this in the following theorem.

**Theorem 2.10.** Suppose \( x(\tau) \) solves \( x'(\tau) = F(x(\tau)) \), \( x(0) = x_0(s) \) with \( F \) continuously differentiable. Then \( \xi = \partial x / \partial s \) solves

\[
\frac{d\xi}{d\tau} = DF(x(\tau))\xi, \\
\xi(0) = x'_0(s).
\]

**Corollary 2.11.** Under the assumptions of Theorem 2.6 the solutions to (2.7) are continuously differentiable with respect to \( s \) provided \( x_0(s), y_0(s) \) and \( u_0(s) \) are continuously differentiable with respect to \( s \).
A proof of this theorem can be found in Brauer and Nohel. Notice the curve \( \Gamma \) and the data \( u_0 \) must be smooth to apply the theorem. Moreover, since the equation for \( \xi \) is linear, existence of solutions is not an issue.

**Example 2.12.** Consider the Lotka-Volterra equations (in the form \( x' = F(x) \))

\[
x' = x - xy
\]

\[
y' = xy - y
\]

with \( x(0) = x_0(s) \) and \( y(0) = 0 \). One checks the solution is \( x = (x_0(s)e^t, 0) \). The differential of the vector field is

\[
DF = \begin{pmatrix}
1 - y & -x \\
y & x - 1
\end{pmatrix}.
\]

As in the previous example \( \xi(t) = \partial x(t)/\partial s, \eta(t) = \partial y(t)/\partial s \) solve

\[
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix}' = \begin{pmatrix}
1 & -x_0(s)e^t \\
0 & x_0(s)e^t - 1
\end{pmatrix}\begin{pmatrix}
\xi \\
\eta
\end{pmatrix}
\]

with \( (\xi(0), \eta(0)) = (x'_0(s), 0) \). Suppose we are interested in \( \xi = \partial x/\partial s \). The solution can be found and is \( (\xi(t), \eta(t)) = (x'_0(s)e^t, 0) \). Which can be found directly by computing \( \partial x/\partial s \) from above.

Before putting this together we note the Jacobian has a nice geometric interpretation in two space variables. The Jacobian can be written

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
x_0(s)e^t \\
x_0(s)e^t - 1
\end{pmatrix}.
\]

Thus a non vanishing Jacobian at \( \tau = 0 \) requires the characteristic curves not to be tangent to \( \Gamma \). We cannot apply the inverse function theorem if they are.

**Theorem 2.13.** *(Fundamental Existence Theorem)* We suppose the following assumptions on the given data \( a, b, c, u_0 \) and the curve \( \Gamma \).

(i) \( \Gamma \) is a smooth, connected, non closed curve (its parameterization \( x_0(s), y_0(s) \) for \( s \in I \) is injective and continuously differentiable on an interval \( I \)).

(ii) The data \( u_0 \) is a continuously differentiable function on \( \Gamma \).

(iii) The functions \( a(x, y, u), b(x, y, u), \) and \( c(x, y, u) \) are continuously differentiable in \( \Omega \) (described at the beginning of the section).

(iv) The characteristics are not tangent to the curve \( \Gamma \). That is, for \( s \in I \)

\[
\begin{vmatrix}
a(x_0(s), y_0(s), u_0(s)) & x'_0(s) \\
b(x_0(s), y_0(s), u_0(s)) & y'_0(s)
\end{vmatrix} \neq 0.
\]

Then a neighborhood of \( \Gamma \) exists so that the Cauchy problem

\[
a(x, y, u)x + b(x, y, u)y = c(x, y, u)
\]

\[
|\Gamma = u_0,
\]

has a unique continuously differentiable solution in that neighborhood.

**Proof.** We just need to check our algorithm for finding the solution. Solving the system of ODEs, Theorem 2.6 gives unique \(x(\tau, s), y(\tau, s)\) and \(u(\tau, s)\) such that \(x(0, s) = x_0(s), y(0, s) = y_0(s),\) and \(u(0, s) = u_0(s)\). The conditions on the characteristics ensures the inverse function theorem applies and the inverse \(\tau = \tau(x, y), s = s(x, y)\) is continuously differentiable near \(\Gamma\). We also note that

\[
s = s(x_0(s), y_0(s)) \quad 0 = \tau(x_0(s), y_0(s)).
\]

We set \(u = u(x, y) := u(\tau(x, y), s(x, y))\). We need only check this solves the PDE and is unique. On \(\Gamma\)

\[
u(x_0(s), y_0(s)) = u(\tau(x_0(s), y_0(s)), s(x_0(s), y_0(s))) = u(0, s) = u_0(s),
\]

and \(u(x, y)\) satisfies the Cauchy data (the initial condition). To show it satisfies the PDE, we apply the chain rule to find

\[
(2.9) \quad u_x = u_\tau \tau_x + u_s s_x, \quad u_y = u_\tau \tau_y + u_s s_y.
\]

Applying the chain rule to \(\tau\) and \(s\) we get

\[
1 = \tau_{\tau} = \tau_x \frac{dx}{d\tau} + \tau_y \frac{dy}{d\tau},
\]

and the characteristic equations then gives

\[
1 = a(x, y) \tau_x + b(x, y) \tau_y.
\]

Similarly

\[
0 = s_{\tau} = s_x \frac{dx}{d\tau} + s_y \frac{dy}{d\tau} = a(x, y) s_x + b(x, y) s_y.
\]

Thus, returning to (2.9),

\[
a(x, y) u_x + b(x, y) u_y = (a \tau_x + b \tau_y) u_\tau + (a s_x + b s_y) u_s = u_\tau = c(x, y, u),
\]

and \(u\) solves the PDE.

Finally we must show the solution is unique. Let’s suppose \(v(x, y)\) solves the PDE and satisfies the Cauchy data. For each \(s \in I\) consider the ODE

\[
\frac{dx}{d\tau} = a(x, y, v(x, y)), \quad \frac{dy}{d\tau} = b(x, y, v(x, y))
\]

\[
x(0) = x_0(s), \quad y(0) = y_0(s).
\]

By the theory of ODEs, Theorem 2.6, this has a solution for \(|\tau|\) small (maybe smaller than before). We set \(v(\tau) = v(x(\tau), y(\tau))\). Then

\[
v'(\tau) = v_x \tau' + v_y \tau' = a v_x + b v_y = c,
\]

and \(v(0) = u_0(s)\). Thus \(v\) satisfies the same system of ODE \(u\) does. Since the solution to (2.7) is unique, we must have \(v = u\). \(\square\)
2.6. Examples with Trouble

The next examples illustrate of the trouble that can occur when the characteristics become tangent to $\Gamma$.

**Example 2.14.** Consider the Cauchy problem (Burgers equation)

$$u_t + uu_x = 1$$
$$u(x, x) = c.$$

**a)** For which $c$ does the existence theorem guarantee a solution near the line $x = y$? Find the solution.

**b)** Show solutions are not unique when $c = 1$. Why does this not contradict the existence theorem?

**Solution.** We start by naively applying our algorithm for finding the solution. To parameterize $\Gamma$ we choose $x_0(s) = s = y_0(s)$ for $s \in \mathbb{R}$. We need to solve

$$\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = u, \quad \frac{du}{d\tau} = 1$$
$$t(0) = s, \quad x(0) = s, \quad u(0) = c.$$

The solution is easily found to be

$$t = \tau + s, \quad x = \frac{\tau^2}{2} + c\tau + s, \quad u = \tau + c.$$

The characteristics are given by

$$x = \frac{(t-s)^2}{2} + c(t-s) + s$$

and are parabolas. Note that the Jacobian at $\tau = 0$ is

$$\begin{vmatrix} t & t_s \\ x & x_s \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \tau + c & 1 \end{vmatrix} = 1 - c.$$

Thus we may expect trouble when $c = 1$ - the parabolas are tangent to the line $x = t$.

To invert we find, from the solution for $u$, $\tau = u - c$. Thus $s = t - u + c$. Inserting these into the equation for $x$ we get

$$x = \frac{(u-c)^2}{2} + c(u-c) + c - u + t$$
$$= \frac{(u-1)^2}{2} + \frac{(c-1)^2}{2} + t.$$

Thus

$$\frac{(u-1)^2}{2} = \frac{(c-1)^2}{2} + x - t.$$

To determine which root to take we go back to equation for $u$ in terms of $\tau$. Note $u - 1 = \tau = c - 1$. If $c > 1$, then $u - 1 > 0$ for $|\tau|$ small. Similarly, if $c < 1$, $u - 1 < 0$ for small $|\tau|$. This tells us which root to take. Therefore,

$$u(t, x) = \begin{cases} 1 + \sqrt{2} \left( x - t + \frac{(c-1)^2}{2} \right)^{1/2} & c > 1 \\ 1 - \sqrt{2} \left( x - t + \frac{(c-1)^2}{2} \right)^{1/2} & c < 1. \end{cases}$$
So the answer to Part a) is $c = 1$. If $c = 1$, then $\tau = u - 1$. If $\tau > 0$, we take the positive root and the negative root otherwise. Thus both

$$ u(x, t) = 1 - \sqrt{2\sqrt{x} - t}, \quad u(x, t) = 1 + \sqrt{2\sqrt{x} - t} $$

are solutions to the PDE and the solution is not unique. This does not contradict the existence theorem since the Jacobian vanishes and the theorem does not apply.
2.7. Homework

Solve the given Cauchy Problem and sketch the characteristics.

1. \( u_x + u_y + u = 1, u = \sin x \) on \( y = x + x^2, x > 0 \).
2. \( xu_x + (x^2 + y)u_y + \left( \frac{2}{x} - x \right) u = 1, u = 0 \) on \( x = 1 \).
3. \( u_x + 3y^2/3u_y = 2, u(x,1) = 1 + x \).
4. \( u_x - 2u_y = u, u = s \) when \( x = 0, y = s \).
5. \( xu_x + yu_y = 2u, u = s^2 \) when \( x = s, y = 1 \).
6. \( yu_x - xu_y = 2xyu, u = s^2 \) when \( x = y = s, s > 0 \).
7. \(-2xyu_x + 4xu_y + yu = 4xy \) when \( u = 2^{1/2} \) on \( y = 0, x > 0 \). In what part of the half plane, \( x > 0 \) is \( u \) determined if its value on the \( x \)-axis is only known for \( 1 \leq x \leq 3 \)?
8. \( yu_x + u_y = x, u = \frac{2}{3} s^3 \) when \( x = s^2, y = s, 1 < s < 2 \).
9. For an arbitrary differentiable function \( f(x) \), the equation \( x^2u_x + 2xyu_y = xu \), cannot have a solution such that \( u = f(x) \) when \( y = 4x^2 \). Find a form for \( f \) so that a solution may be found, and for such \( f \) show that the solution is not unique. Why does this not contradict the Fundamental Existence Theorem?
10. \( xu_x - yu_y = u, u = s^2 \) when \( x = y = s, 1 \leq s \leq 2 \). State where your solution is defined. For an arbitrary differentiable function \( f(s) \), the equation has no solution equal to \( f(s) \) in the curve \( x = s, y = 1/s \) \( s > 0 \). Determine the general form of \( f \) when a solution exists and show that, in this case, there is no unique solution.
11. In \( \mathbb{R}^3 \) solve \( xu_x + 2yu_y + u_z = 3u \) with \( u = \phi(x,y) \) on the plane \( z = 0 \).
12. \( u_x + u_y = u, u = \cos x \) when \( y = 0 \).
13. \( x^2u_x + y^2u_y = u^2, u = 1 \) when \( y = 2x \).
14. \( u_x + 2u_y = u^2, uu(x,0) = h(x) \).
15. In \( \mathbb{R}^3 \) solve \( u_x + xu_y - u_z = u \) with \( u(x,y,1) = x + y \).
16. \( xu_x + u_y = y, uu(x,0) = x^2 \).
17. \( u_x - 2u_y = u, uu(0,y) = y \).
18. \( y^{-1}u_x + u_y = u^2, uu(x,1) = x^2 \).
19. \( xu_x + yu_y + uz = u \) with \( uu(x,y,0) = h(x,y) \).
20. \( u_x + u_y + zu_z = u^3 \) with \( uu(x,y,1) = h(x,y) \).
21. \( xu_x + yu_y = x \) with \( u = 0 \) on the circle \( x^2 + y^2 = 1 \).
22. \( yu_x - xu_y = xyu \) with \( uu(x,x) = 1 \).
23. \( u_x + 2u_y = 1 - u \) with \( u = 0 \) on \( y = -1/x \) and for \( x > \sqrt{\frac{1}{2}} \).
24. \( (x + y)u_x + (x + y)u_y = 2u, uu(x,0) = e^x \).
25. \( u_x + 2u_y = 1 - u, uu = 0 \) on \( y = 1/x, x > 0 \). The solution is defined on the whole \( x - y \) plane, but it is not differentiable at \( (0,0) \).
26. \( x^2u_x - xyu_y + y^2 = 0 \), with \( u = 2s^3 \) on \( x = s, y = s^2, 1 \leq s \leq 2 \). State the region in which your solution is defined.
27. \( (y + u)u_x + yu_y = x - y \) when \( u = 1 + x \) on the curve \( y = 1 \).
28. \( (y^2 - u^2)u_x - xyu_y = xu \) containing \( x = y = u, x > 0 \).
29. \( uu + uu_x = 1 \) with \( u(2s^2, 2s) = 0, s > 0 \).
30. \( (y - u)u_x + (u - x)u_y = x - y; uu = 0 \) when \( y = 2x \).
31. \( (2xy + 2y^2 + u)u_x - (2x^2 + 2xy + u)u_y = 2u(x - y), uu = 2x^2 \) when \( y = 0 \).
32. \( (3y - 2u)u_x + (u - 3x)u_y = 2x - y; uu = 0 \) when \( x = y \).
33. \( xu_x + y^2u_y = u^2, uu = s \) when \( x = 1/s, y = 2s, s > 0 \).
34. \( x^2u_x + uu_y = 1; \ u = 0 \) when \( x = s, y = 1 - s \).
35. \( u_y + uu_x = 1, \ u = s/2 \) when \( x = y = s, \ 0 \leq s \leq 1 \).
36. Show that the PDE \( u_y + uu_x = 1 \) has no solution such that \( u = \frac{1}{2}y \) when \( 4x - y^2 = 0 \), and has no unique solution such that \( u = y \) when \( 2x - y^2 = 0 \). Why does this not contradict the Fundamental Existence Theorem?
37. Determine in parametric form the solution of \( u_y + uu_x + \frac{1}{2}u = 0 \) with \( u(x,0) = \sin x \). Show that there exists a single-valued differentiable solution provided that \( y < 2 \log 2 \).
38. Show that the general solution of \( xu_x - yu_y = 0 \) is \( u = f(xy) \). Find the solution whose graph contains the line \( u = x = y \) (the answer is \( u = \sqrt{xy} \)). What happens when in initial curve is \( y = 1/x \)?
39. \( uu_x + yu_y = x \) with \( u(x,1) = 2x \).
40. \( u_x + u^2u_y = 1, \ u(x,0) = 1 \).
41. \( u_x + \sqrt{uu_y} = 0, \ u(x,0) = x^2 + 1 \).
42. Show that the general solution to \( u_x + u_y = \sqrt{u} \) is \( u(x,y) = (x + f(x-y))^2/4 \). Observe that the trivial solution \( u = 0 \) is not covered by the general solution.
43. \( xuu_x - yuu_y = y^2 - x^2; \ u = 1 \) on \( x^2 + y^2 = 1 \).
44. \( (y-u)u_x + (u-x)u_y = x - y, \ u = 0 \) on \( xy = 1 \).
45. \( uu_x = y^2x^2, \ u(x,x) = 1 \).
46. \( u_x + u^2u_y = 0, \ u(x,0) = x \).
47. Find a condition on the initial data \( f \) so that the solution to the damped wave equation \( u_t + uu_x + cu = 0 \) with initial condition \( u(x,0) = f(x) \) has a solution for all \( t > 0 \).
48. \( u_y + uu_x = 1 \) with \( u(s,s) = c \). Find the unique solution if \( c \neq 1 \). If \( c = 1 \), show that the solution is not unique. Explain.
49. Show that \( (u-1)u_x + u_y = u \) with \( u(0,y) = 1 \) has at least two solutions for \( x \geq 0 \), but no solution for \( x \leq 0 \). Why does this fact not contradict the local existence and uniqueness result?
Selected Answers:
1. \(x = s + t, y = s + s^2 + t, u = 1 - e^s(1 - \sin s)e^{-(s+t)} = 1 + e^{-x}(\sin(y - x)^{1/2} - 1)e^{(y-x)/x}, y \geq x.
2. \(x = e^t, y = (s - 1)e^t + e^{2t}, u = \frac{1}{s-1}(1 - e^{-(s-1)t}) = \frac{x}{y-x}(1 - x^{-y-x^2}/x).
3. \(x = s + t, y = (t + 1)^3, u = 2t + 1 + s = x + y^{1/3}.
4. \(x = t, y = s - 2t, u = se^t = (y + 2x)e^t.
5. \(x = se^t, y = e^t, u = s^2e^{2t} = x^2.
6. \(x = s(\cos t + \sin t), y = s(\cos t - \sin t), u = s^2e^2\sin 2t = \frac{1}{2}(x^2 + y^2)e^{\frac{1}{2}(x^2-y^2)}.
7. \(u = -4x + 2\sqrt{x} + 2\sqrt{y}(y^2 + 4x)^{1/2} for x, y between the parabolas y^2 + 4x = 4 and y^2 + 4x = 12.
8. \(x = st + \frac{1}{2}t^2 + s^2, y = s + t, u = \frac{1}{6}(t^3 + 3st^2 + 6s^2t + 4s^3) = xy - \frac{1}{3}y^3, 1 < 2x - y^2 < 4.
9. Differentiate \(u(x, 4x^2) = f(x) and show an inconsistency unless \(f = cx). Two solutions are cx, cy/(4x).
10. \(x = se^t, y = se^{-t} u = s^2e^t = x^{3/2}y^{1/2} between the hyperbolas xy = 1 and xy = 4. Differentiating \(u(s, 1/s) = f produces an inconsistency with the differential equation unless \(f = cs). Two solutions are cx and cy/(4x).
11. \(x = s_1e^t, y = s_2e^{2t}, z = t, u = (s_1, s_2)e^{3t} = \phi(x, y, z)e^{-z}, ye^{-x^3}e^{3z}.
12. \(u = e^t \cos(x - y).
13. \(u = xy/(xy - y + 2x).
14. \(x = t + s, y = 2t, u = \frac{h(x)}{1-h(s)} = \frac{h(x-y)}{1 - \frac{x}{y}}.
15. \(x = t + s, y = \frac{1}{2}t^2 + s + s^2, z = -t + 1 u = (s_1 + s_2)e^t = (x + y + (z - 1)(1 + x + \frac{1}{2}(z - 1)^2))e^{-z}.
16. \(u = \frac{1}{6}y^2 + x^2e^{-2y}.
17. \(u = h(xe^{-z}, ye^{-z})e^z.
18. \(x = e^t \cos s, y = e^t \sin s, u = e^t \cos s - \cos s = x - \frac{x}{2x + y^2 + 2t}.
19. \(u = \exp \left(\frac{1}{2}(x^2 - y^2)\right).
20. \(x = t + s, y = 2t - 1/s, u = 1 - e^{-t} = 1 - \exp \left[-\frac{1}{2}(2x + y) - (2x - y^2 - 8)^{1/2}\right].
21. \(x = s/(1 - st), y = s^2/(1 - st), u = \frac{s^2}{16(x - s)^3} + \frac{3s}{8} = \frac{y^2}{16} + \frac{3}{2}xy, 1 < xy < 8, x > 0.
22. \(x = (s + 1)e^t - e^{-t}, y = e^t, u = se^t - e^{-t} = \frac{x}{y} + (x - y), y > 0.
23. \(x^2 = -(s - t)^2 + 3s^2 - \frac{s^4}{(s-t)^2}, y = s - t, u = \frac{s^2}{s-t} = \frac{3y^2 - 2y^2 + 2t^2}{2y^2}.
24. \(x = \frac{1}{2}t^2 + 2s^2, y = 2s + t, u = t = \frac{1}{2}y - (x - \frac{1}{4}y^2)^{1/2}, x > \frac{1}{4}y^2.
25. \(5(x + y + u)^2 = 9(x^2 + y^2 + u^2).
26. \(4(x^2 + y^2 - u^2) = -u^2(x^2 + y^2).
27. \(\sqrt{3}(x + 2y + 3u) = 3(x^2 + y^2 + u^2)^{1/2}.
28. \(s(1 - st)x = 1, (1 - 2st)y = 2s, (1 - st)u = s; y^2xu = 4(y - u)^2.
29. \((1 - st)x = s, y = \frac{1}{2}t^2 + (1 - s), u = t; (1 + xu)(2 - 2y + u^2) = 2x.
30. \(x = \frac{1}{2}t(t + s) + s, y = t + s, u = \frac{1}{2}s + t = \frac{2x + yu^2}{2(y - u^2)}.
31. \(u \left(\frac{y^2}{16}, y\right) = \frac{1}{2}y, \frac{1}{2}yu + uy = \frac{1}{2} = uu + uy, which gives the contradiction. Two solutions are y, \sqrt{2x}.
32. \(x = 2 \sin(s)(1 - e^{-t/2}) + s, y = t, u = \sin(se^{-t/2}). The Jacobian vanishes when 2(1 - e^{-t/2}) \cos s = -1 and no solutions for s when t < 2 \log 2. For t > 2 \log 2, x_s = 0 has, for any t, a solution s = s_0 and multi-valuedness.
39. \( x = \frac{3}{2}se^t - \frac{1}{2}se^{-t}, \ y = e^t, \ u = \frac{3}{2}se^t - \frac{1}{2}se^{-t} = x^{\frac{3y^2+1}{3y^2-1}}. \)

40. \( u = (3y + 3)^{1/3}. \)

43. \( x = e^t \cos s, \ y = e^{-t} \sin s, \ u = (2 - x^2 - y^2)^{1/2}. \)

44. \( u = \left(3y + 3\right)^{1/3}. \)

45. \( x = e^t \cos s, \ y = e^{-t} \sin s, \ u = \left(2 - x^2 - y^2\right)^{1/2}. \)

47. \( f'(t) > -c. \)

More Solutions

36. Consider the PDE \( u_y + uu_x = 1. \)

(a) Show the PDE has no solution such that \( u = \frac{1}{2}y \) when \( 4x - y^2 = 0. \) Do this by differentiating \( u(y^2/4, y) \) and comparing it with the PDE evaluated on the curve \( x = y^2/4. \)

(b) Show the PDE has no unique solution such that \( u = y \) when \( 2x - y^2 = 0. \) Do this by applying our algorithm for finding the solution. At some stage you will have two choices for \( u. \) Check that both solve the PDE and satisfy the data.

(c) Why does the results in the two previous parts not contradict the Fundamental Existence Theorem?

Solution. We parameterize the curve \( 4x - y^2 = 0 \) by letting \( y = s \) and \( x = s^2/4. \) We are told \( u(s^2/4, s) = \frac{1}{2}y = \frac{1}{2}s. \) Differentiating we find

\[
\frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial y} \frac{dy}{ds} = u_x \frac{s}{2} + u_y 1 = \frac{1}{2}.
\]

However, if we evaluate the PDE on the curve \( \Gamma, \) we find

\[
u_y + \frac{1}{2}su_x = 1.
\]

Comparing the two we conclude \( 1 = \frac{1}{2}, \) and so the solution does not exist.

To see what went wrong, we compute the Jacobian

\[
|J| = \begin{vmatrix} x_x & x_s \\ y_x & y_s \end{vmatrix}_{s=0} = \begin{vmatrix} \frac{1}{2}s & \frac{1}{2}s \\ 1 & 1 \end{vmatrix} = 0,
\]

and so the characteristics are tangent to \( \Gamma \) everywhere and the fundamental existence theorem does not apply.

For (b) we attempt to solve the PDE the usual way. We parameterize \( \Gamma \) by setting \( y = s \) and \( x = s^2/2. \) Then

\[
\frac{dx}{ds} = u, \quad \frac{du}{ds} = 1, \quad u(0, s) = s.
\]

Solving, in reverse order, we find

\[
u(\tau, s) = \tau + s, \quad x(\tau, s) = \tau^2/2 + \tau s + s^2/2 = \frac{1}{2}(\tau + s)^2.
\]
We see we can choose \( u = y \) or \( u = \sqrt{2x} \) or \( u = 2x/y \). Again the problem is that the Jacobian vanishes. Indeed
\[
|J| = \begin{vmatrix} \tau + s & \tau + s \\ 1 & 1 \end{vmatrix} = 0,
\]
so the fundamental existence theorem does not apply.

9. Consider the PDE \( x^2u_x + 2xyu_y = xu \) with \( u(x,4x^2) = f(x) \).

(a) Find a form for \( f \) such that if the form is not satisfied, the PDE has no solution. Do this by simply differentiating \( u(x,4x^2) = f(x) \) and comparing it to the PDE. This will provide an ODE \( f \) must satisfy. Solve it to find the form for \( f \).

(b) If \( f \) does have this the form found in Part (a), show the solution to the PDE is not unique. Copy Part (b) above for this.

(c) Why does the results in the two previous parts not contradict the Fundamental Existence Theorem?

Solution. This is nearly the same as the previous problem. We start by parameterizing \( \Gamma \) with \( x = s \) and \( y = 4s^2 \). The data implies
\[
u(s,4s^2) = f(s).
\]
Differentiating we get
\[
u_x 1 + u_y 8s = f'(s).
\]
Next we express the PDE on the curve \( \Gamma \). We find, dividing by \( x \) first,
\[
su_x + 8s^2u_y = u = f(s)
\]
or
\[
s(u_x + 8su_y) = f(s).
\]
Comparing the two we conclude that a necessary condition for the solution to exist is \( sf' = f \). This is easy to solve and we find
\[
f(s) = c_0s,
\]
where \( c_0 \) is any constant.

If \( f(x) = c_0x \), we attempt to solve the equations in the usual manner. We have
\[
\begin{align*}
\frac{dx}{d\tau} &= x, \quad x(0, s) = s \\
\frac{dy}{d\tau} &= 2y, \quad y(0, s) = 4s^2 \\
\frac{du}{d\tau} &= u, \quad u(0, s) = c_0s
\end{align*}
\]
Solving we find
\[
x(\tau, s) = se^\tau, \quad y(\tau, s) = 4s^2 e^{2\tau}, \quad u(\tau, s) = c_0se^\tau.
\]
We see we can choose \( u = c_0x \) or \( u = c_0\sqrt{y}/2 \) or \( u = c_0y/4x \). Again the problem is that the Jacobian vanishes. Indeed
\[
|J| = \begin{vmatrix} se^\tau & e^\tau \\ 8s^2e^{2\tau} & 8se^{2\tau} \end{vmatrix} = 0,
\]
so the fundamental existence theorem does not apply.

49. Consider \( (u-1)u_x + u_y = u \) with \( u(0,y) = 1 \). The goal here is to show the PDE has at least two solutions for \( x \geq 0 \), but no solution for \( x < 0 \).
(a) Show that the solution, if it exists, satisfies $x = u - \ln u - 1$ with $u > 0$.

(b) Use the inequality $\ln(u) + 1 \leq u$ for all $u > 0$ show there is no solution for $x < 0$.

(c) We need to know, given $x = f(u)$, if we can solve for $u = f^{-1}(x)$ even if we cannot find an explicit formula for the inverse. Apply the following theorem from MAT 371:

Suppose the function $x = f(u)$ with $f: [a, b] \to \mathbb{R}$ is continuous and differentiable with $f'(u) \neq 0$ on $(a, b)$. Then $u = f^{-1}(x)$ exists and is continuous. Moreover, if $c \in [a, b]$ and $f'(c) \neq 0$, then $f^{-1}$ is differentiable at $d = f(c)$.

Apply the theorem to the intervals $[1/2, 1]$ and $[1, 2]$ to conclude the PDE has two different solutions. You might graph the function $f(u) = u - \ln u - 1$ on $[1, 5]$ to see what’s going on.

**Solution.** We attempt to solve in the usual way. We parameterize $\Gamma$ by $x = 0$ and $y = s$. We find

\[
\frac{dx}{\tau} = u - 1 \quad \frac{du}{\tau} = 1 \quad \frac{ds}{\tau} = u
\]

Solving for $y$ and $u$ first, we find

\[
y(\tau, s) = \tau + s, \quad u(\tau, s) = e^\tau, \quad x(\tau, s) = e^\tau - \tau - 1.
\]

From the solutions for $u$ and $x$ we discover

\[
x = u - \ln u - 1.
\]

Unfortunately we are not smart enough to invert this. We have to figure another way. From the mean-value theorem we have

\[
\ln(u) + 1 \leq u
\]

for all $u > 0$. This implies

\[
0 \leq u - \ln(u) - 1
\]

This means there is no solution to $x = u - \ln u - 1$ for $x < 0$. It is worth graphing $x = u \ln u - 1$ to see what’s going on. For positive $x$ one can see there are two values of $u$. We need to know the inverse of each branch is a differentiable function.

So just apply the given theorem and check. We set $f(u) = u - \ln u - 1$. Then $f'(u) = 1 - \frac{1}{u}$. This is not zero on each interval $[1/2, 1]$ and $(1, 2]$. So the theorem applies and $f^{-1}(x) = u$ exists and is differentiable. By construction it solves the PDE and since it take on different values of $u$ on each interval, the two solutions it generates are not the same. \qed