A Collection of Nice Problems

1a. Suppose \( \int_{-\pi}^{\pi} (f(x))^2 dx < \infty \). By starting with the obvious inequality \( 0 \leq \int_{-\pi}^{\pi} (f(x) - S_n(x))^2 dx \) prove Bessel’s inequality,
\[
\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, dx .
\]

1b. Use Bessel’s inequality and the \( n^{th} \) term test for infinite series to prove the following version of the Riemann Lebesgue Theorem: Suppose \( \int_{-\pi}^{\pi} (f(x))^2 dx < \infty \). Then
\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} \cos(nx)f(x) \, dx = 0 = \lim_{n \to \infty} \int_{-\pi}^{\pi} \sin(nx)f(x) \, dx .
\]

2. Suppose a function, \( f \), has the Fourier series
\[
f(x) := \sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}
\]
Using a test called Dirichlet’s test one can show that the infinite series converges for all \( x \) (assume this). Show that \( f(x) \) cannot be piecewise continuous.

3. Use Problem 7 in Section 3.2 and Dirichlet’s theorem to prove the following theorem:

If \( f(x) \) is continuous and periodic, \( f'(x) \) piecewise smooth, then the Fourier series of \( f(x) \) may be differentiated. Furthermore, the differentiated series converges to \( f'(x) \) everywhere \( f''(x) \) is continuous.

Notice that you do not need the concept of uniform convergence in this proof.

4. Suppose that \( g(x) \) is defined on \( 0 \leq x \leq L \) and that \( g(x) \) is continuous. Let \( f_e(x) \) be the even periodic extension of \( g(x) \). Use the theorem proven in Problem 3 to prove the following result.

Suppose that \( f'_e(x) \) piecewise smooth. Then the Fourier series of \( f_e(x) \) may be differentiated. Furthermore, the differentiated series converges to \( f'_e(x) \) everywhere \( f''_e(x) \) is continuous.
5. Suppose that \( g(x) \) is defined on \( 0 \leq x \leq L \) and that \( g(x) \) is continuous. Let \( f_o(x) \) be the odd periodic extension of \( g(x) \). Use the theorem proven in Problem 3 to prove the following result.

Suppose that \( f'_o(x) \) is piecewise smooth and that \( f(0) = f(L) = 0 \). Then the Fourier series of \( f_o(x) \) may be differentiated. Furthermore, the differentiated series converges to \( f'_o(x) \) everywhere \( f''_o(x) \) is continuous.

6. Suppose that \( |a_n| \leq Mn^{-3.5}, \ |b_n| \leq Mn^{-3.5} \) for \( n \geq 1 \). How many continuous derivatives must the function defined by

\[
f(x) = \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx
\]

possess?

7. Suppose that \( f \) is continuous, \( f' \) is piecewise continuous, and \( g \) is continuously differentiable on \([a, b]\). Verify the integration by parts formula

\[
\int_{a}^{b} fg' = fg|_{a}^{b} - \int_{a}^{b} f'g.
\]

Recall \( \int_{a}^{c} f = \lim_{x \to c^-} \int_{a}^{x} f \)