Elliptic PDE Problems

1. Let \( \Omega = \{(x, y) \in \mathbb{R}^2 : 0 < y < 1, x > 0\} \). Find the bounded solution to
\[
\nabla^2 u = 0 \quad 0 < x < \infty, \quad 0 < y < 1, \n
\]
\[
u(0, y) = f(y), \quad 0 < y < 1, \n
\]
\[
u(x, 0) = 0, \quad 0 < x, \n
\]
\[
u(x, 1) = 0, \quad 0 < x. \n
\]

Separation of variables will work for this problem even though \( \Omega \) is unbounded. Is your solution \( C^\infty(\Omega) \cap C(\overline{\Omega}) \)? Prove or disprove (be careful—the proof of Theorem D required a finite interval).

2. Find the solution to Laplace’s equation on the rectangle \( 0 < x < 1, 0 < y < 1 \) for each for the following sets of boundary conditions:

(a) \( u_x(0, y) = 0; \ u = 1 \) on the remainder of the boundary.

(b) \( u_x(0, y) = u_x(1, y) = 0; \ u(x, 0) = 0 \) and \( u(x, 1) = 1. \)

Answer:

(a) \[
u(x, y) = \sum_{n=1}^\infty a_n \frac{\sinh(\lambda_n y) + \sinh(\lambda_n(1-y))}{\sinh(\lambda_n)} \cos(\lambda_n x) + \sum_{n=1}^\infty b_n \frac{\cosh(\mu_n x)}{\cosh(\mu_n)} \sin(\mu_n y), \]
where \[
\lambda_n = \frac{(2n - 1)\pi}{2}, \quad a_n = \frac{4(1)^n}{\pi(2n - 1)}, \quad \mu_n = n\pi, \quad b_n = 2\frac{(1 - (-1)^n)}{n\pi}. \]

(b) \( u(x, y) = y \)

3. Prove that the solution to Poisson’s equation is unique. Hint: suppose
\[
\nabla^2 u = f(x), \quad x \in \Omega, \n
\]
\[
u|_{\partial \Omega} = g(x), \quad x \in \partial \Omega \n
\]
has two solutions, \( u_1, u_2 \). Set \( w = u_1 - u_2 \). Construct the PDE \( w \) solves, multiply this PDE by \( w \), integrate, and use the divergence theorem.
4. Solve Laplace’s equation on the quarter disk \(0 < r < 1, 0 \leq \theta < \pi/2\), with \(u_\theta(r, 0) = 0\), \(u(r, \pi/2) = 0\) for \(0 < r < 1\) and \(u(1, \theta) = f(\theta)\) for \(0 \leq \theta < \pi/2\).

Answer:

\[
u(r, \theta) = \sum_{n=1}^{\infty} a_n r^{2n-1} \cos((2n - 1)\theta),\]

where \(a_n = \frac{4}{\pi} \int_{0}^{\pi/2} f(\theta) \cos((2n - 1)\theta) \, d\theta\).

5. Solve Laplace’s equation on the semi disk \(0 < r < 1, 0 \leq \theta < \pi\) with \(u(1, \theta) = g(\theta)\), \(u(r, 0) = 0\), and \(u(r, \pi) = 0\) for \(0 < r < 1\).

Answer:

\[
u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin(n\theta),\]

where \(b_n = \frac{2}{\pi} \int_{0}^{\pi} g(\theta) \sin(n\theta) \, d\theta\).

6. Solve Laplace’s equation on the annulus \(a < r < b, -\pi \leq \theta < \pi\) with \(u(a, \theta) = g(\theta)\) and \(u(b, \theta) = f(\theta)\) for \(-\pi \leq \theta < \pi\).

Answer: A bit messy. Enjoy it!