**Definitions List**

**Definition of Supremum:** Let $S \subset \mathbb{R}$ be non empty. The *Supremum* of $S$, $\alpha$, and written $\alpha = \sup S$ (sometimes denoted $\alpha = \operatorname{lub} S$ for least upper bound), has the following properties.

(i) The element $\alpha$ is an upper bound of $S$. That is, $x \leq \alpha$ for all $x \in S$.

(ii) The element $\alpha$ is the least upper bound of $S$. That is, $\alpha \leq M$ for any other upper bound, $M$, of $S$.

**Definition of Sequence:** A *Sequence* is a function whose domain is the natural numbers.

**Definition of Sequence Converging:** A sequence of real numbers, $\{a_n\}_{n \in \mathbb{N}}$, converges to a real number, $a$, if and only if, for each $\epsilon > 0$, there exists a natural number $N$, such that, for all $n \geq N$, $|a_n - a| < \epsilon$. When $\{a_n\}_{n \in \mathbb{N}}$ converges to $a$ we write $\lim_{n \to \infty} a_n = a$ or $a_n \to a$ as $n \to \infty$.

**Definition of an Open Set:** A subset $O$ of $\mathbb{R}$ is called *open* in $\mathbb{R}$ if and only if, for each point $x \in O$, there is a $r > 0$ such that all points $y$ in $\mathbb{R}$ satisfying $|x - y| < r$ also belong to the set $O$.

**Definition of a Neighborhood...**

(a) A subset of $\mathbb{R}$ is called *closed* in $\mathbb{R}$ if its complement is open in $\mathbb{R}$.

(b) A *neighborhood* of $x \in \mathbb{R}$ is any set containing an open set containing $x$.

(c) A point $x \in \mathbb{R}$ is called a *boundary point* of a set $S \subset \mathbb{R}$ if every neighborhood of $x$ contains a point in $S$ and a point not in $S$.

(d) A point $x \in \mathbb{R}$ is called an *interior point* of a set $S \subset \mathbb{R}$ if there exists a neighborhood of $x$ contained in $S$.

(e) A point $x \in \mathbb{R}$ is called an *exterior point* of a set $S \subset \mathbb{R}$ if there is a neighborhood of $x$ which is entirely contained in complement of $S$.

**Definition of Subsequence:** Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence and $\{n_k\}_{k \in \mathbb{N}}$ be any sequence of natural numbers such that $n_1 < n_2 < n_3 < \ldots$. The sequence $\{a_{n_k}\}_{k \in \mathbb{N}}$ is called a *subsequence* of $\{a_n\}_{n \in \mathbb{N}}$.

**Definition of Cluster Point:** Let $E \subset \mathbb{R}$. A point $a \in \mathbb{R}$ is called a *cluster point* or *accumulation point* of $E$ if and only if every neighborhood of $a$ contains a point in $E$ distinct from $a$.

**Definition of a Cauchy Seq:** A sequence $\{x_n\}_{n \in \mathbb{N}}$ is called *Cauchy* (pronounced Co She) if and only if for all $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ for all $n, m \geq N$.

**Definition of a Function:** A *function* from $D$ into $\mathbb{R}$, denoted $f : D \to \mathbb{R}$, is a subset of $D \times \mathbb{R}$ with the property that for each $x \in D$ there is exactly one $y \in \mathbb{R}$ with $(x, y)$ in this subset. We write $f(x) = y$.

**Definition of a Continuous Function:** Suppose $f : D \to \mathbb{R}$.

(i) A function $f$ is *continuous* at $c \in D$ if and only if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$, $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.

(ii) A function $f$ is called continuous if and only if it is continuous at each point in its domain $D$. 
Definition of Limit of a Function: Let \( f : \mathcal{D} \to \mathbb{R} \) with \( x_0 \) a cluster point of \( \mathcal{D} \). Then \( f \) has a limit \( L \) at \( x_0 \) if and only if, for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( x \in \mathcal{D} \), \( 0 < |x - x_0| < \delta \) implies \( |f(x) - L| < \epsilon \). We denote the limit \( \lim_{x \to x_0} f(x) = L \).

Definition of IVP: A function, \( f : [a, b] \to \mathbb{R} \), is said to have the intermediate-value property if and only if, given any \( y_0 \) between \( f(a) \) and \( f(b) \), there exists a \( x_0 \in [a, b] \) with \( f(x_0) = y_0 \).

Definition of Uniformly Continuous: A function \( f : \mathcal{D} \to \mathbb{R} \) is uniformly continuous on \( \mathcal{D} \) if and only if, for all \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( x, a \in \mathcal{D} \), \( |x - a| < \delta \) implies \( |f(x) - f(a)| < \epsilon \).

Definition of the Derivative: Let \( f : \mathcal{D} \to \mathbb{R} \) with \( x_0 \) a cluster point of \( \mathcal{D} \) and \( x_0 \in \mathcal{D} \). We say \( f \) is differentiable at \( x_0 \) if and only if \( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \) exists, in which case we call the limit \( f'(x_0) \).

Definition of a Partition: Let \([a, b]\) be a closed, bounded interval.

(i) A Partition of \([a, b]\) is a set of points \( P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\} \) with \( a = x_0 < x_1 < \cdots < x_n = b \).

(ii) The norm of a partition is \( ||P|| = \max_{1 \leq i \leq n} |x_i - x_{i-1}| \).

(iii) A Refinement of \( P \) is a partition \( Q \) of \([a, b]\) such that \( P \subset Q \). We say \( Q \) is finer than \( P \).

Definition of Upper and Lower Darboux Sums: Let \( f : [a, b] \to \mathbb{R} \) be bounded. Set \( M_j(f) = \sup_{x_{j-1} \leq x \leq x_j} f(x) \) and \( m_j(f) = \inf_{x_{j-1} \leq x \leq x_j} f(x) \).

(a) The upper-Darboux Sum of \( f \) over \( P \) is \( U(P, f) = \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1}) \).

(b) The lower-Darboux Sum of \( f \) over \( P \) is \( L(P, f) = \sum_{j=1}^{n} m_j(f)(x_j - x_{j-1}) \).

Definition of the Darboux Integral: The function \( f : [a, b] \to \mathbb{R} \) is said to be Darboux Integrable on \([a, b]\) if \( f \) is bounded and, for all \( \epsilon > 0 \), there exists a partition, \( P \), of \([a, b]\) such that \( U(P, f) - L(P, f) < \epsilon \). In this case we say \( f \in \mathcal{R}[a, b] \) (the \( \mathcal{R} \) stands for Riemann).

Definition of a Riemann Sum: Let \( P \) be any partition of \([a, b]\) and \( f : [a, b] \to \mathbb{R} \) is bounded. The Riemann Sum with respect to \( P \) is \( S(P, f) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \), where \( x_{i-1} \leq t_i \leq x_i \).