Differentiation

In this chapter we add more structure to the functions we consider. Not surprisingly, we also strengthen our results.

**Definition 5.1.** Let \( f : \mathcal{D} \to \mathbb{R} \) with \( x_0 \) a cluster point of \( \mathcal{D} \) and \( x_0 \in \mathcal{D} \). We say \( f \) is differentiable at \( x_0 \) if and only if

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}
\]

exists, in which case we call the limit \( f'(x_0) \).

**Theorem 5.2.** Suppose \( f : I \to \mathbb{R} \) with \( I \) an interval and \( x_0 \in I \). Then, \( f \) is differentiable at \( x_0 \) if and only if, for all sequences \( \{x_n\}_{n \in \mathbb{N}} \in I \) with \( x_n \neq x_0 \) for all \( n \in \mathbb{N} \) and converging to \( x_0 \), we have the limit

\[
\lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}
\]

exists.

**Proof.** This is the sequential characterization of Limits, Theorem 4.21.

**Example 5.3.** Show the function \( f(x) = |x| \) is not differentiable at the origin.

**Proof.** Consider the sequences

\[
\{x_n\}_{n \in \mathbb{N}} = \left\{ \frac{1}{n} \right\}_{n \in \mathbb{N}}, \quad \{y_n\}_{n \in \mathbb{N}} = \left\{ -\frac{1}{n} \right\}_{n \in \mathbb{N}}.
\]

Note that

\[
\frac{f(x_n) - f(0)}{x_n - 0} = 1, \quad \frac{f(y_n) - f(0)}{y_n - 0} = -1,
\]

for all \( n \in \mathbb{N} \). This implies the limit in Theorem 5.2 does not exist.

As expected, differentiation is stronger than continuity.
Theorem 5.4. Let \( f : I \to \mathbb{R} \) be differentiable at \( x_0 \in I \). Then, \( f \) is continuous at \( x_0 \).

Proof. Set
\[
T(x) = \frac{f(x) - f(x_0)}{x - x_0}.
\]
By assumption \( T \) has a limit at \( x_0 \). Note the function
\[
f(x) = \frac{f(x) - f(x_0)}{x - x_0} + f(x_0) = T(x)(x - x_0) + f(x_0)
\]
also, by the algebra of limits, has a limit at \( x_0 \). Indeed, \( \lim_{x \to x_0} f(x) = f(x_0) \). By Theorem 4.11 \( f \) is continuous at \( x_0 \).

\[\square\]

Theorem 5.5. (Algebra of Derivatives) Suppose \( f, g : I \to \mathbb{R} \) are differentiable at \( x_0 \in I \). Then

(i) \( (f + g)'(x_0) = f'(x_0) + g'(x_0) \),
(ii) \( (fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0) \),
(iii) If \( g(x_0) \neq 0 \), \( \left( \frac{f}{g} \right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)} \).

Proof. We use Theorem 5.2 and the algebra of limits. Let \( \{x_n\}_{n \in \mathbb{N}} \subset I/\{x_0\} \) be any sequence converging to \( x_0 \). By assumption the sequences
\[
\left\{ \frac{f(x_n) - f(x_0)}{x_n - x_0} \right\}_{n \in \mathbb{N}}, \quad \left\{ \frac{g(x_n) - g(x_0)}{x_n - x_0} \right\}_{n \in \mathbb{N}}
\]
converge to \( f'(x_0) \) and \( g'(x_0) \) respectively. Then
\[
(f(x_0) + g(x_0))' = \lim_{n \to \infty} \frac{f(x_n) + g(x_n) - (f(x_0) + g(x_0))}{x_n - x_0}
= \lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} + \lim_{n \to \infty} \frac{g(x_n) - g(x_0)}{x_n - x_0}
= f'(x_0) + g'(x_0).
\]
To show (ii) note
\[
\frac{f(x_n)g(x_n) - f(x_0)g(x_0)}{x_n - x_0} = f(x_n)\frac{g(x_n) - g(x_0)}{x_n - x_0} + g(x_0)\frac{f(x_n) - f(x_0)}{x_n - x_0}.
\]
Moreover, since \( f \) is differentiable at \( x_0 \) it is continuous there. Thus, \( \lim_{n \to \infty} f(x_n) = f(x_0) \). Taking the limit of (5.1) and applying the algebra of limits, we find (ii).

Part (iii) is similar. Since \( g \) is continuous at \( x_0 \) and \( g(x_0) \neq 0 \), the sign preserving property, Lemma 4.28, implies there is an interval, \( I' \subseteq I \), containing \( x_0 \) such that \( g \neq 0 \) on \( I' \). We apply Theorem 4.21 to \( I' \). Then
\[
\frac{f(x_n) - f(x_0)}{g(x_n)} - \frac{f(x_0)}{g(x_0)} = \frac{g(x_0)}{g(x_n)} \frac{f(x_n) - f(x_0)}{x_n - x_0} - \frac{f(x_0)}{g(x_0)} \frac{g(x_n) - g(x_0)}{x_n - x_0}.
\]
As before we may take the limit of both sides and obtain the result.

\[\square\]
We must not forget the chain rule. The proof is based on the one given in [1] and more difficult than one might expect.

**Theorem 5.6. (Chain Rule)** Let \( I \) and \( J \) be intervals with \( f : I \to J \) and \( g : J \to \mathbb{R} \). Let \( c \in I \) with \( f \) differentiable at \( c \) and \( g \) differentiable at \( f(c) \). Then, the composite function \( g \circ f \) is differentiable at \( c \) and

\[
(g \circ f)'(c) = g'(f(c))f'(c).
\]

**Proof.** Let \( y_0 = f(c) \) and define

\[
h(y) = \begin{cases} \frac{g(y) - g(y_0)}{y - y_0} - g'(y_0) & y \neq y_0 \\ 0 & y = y_0. \end{cases}
\]

Since \( g \) is differentiable at \( y_0 \), we see \( \lim_{y \to y_0} h(y) = 0 = h(y_0) \) and so \( h \) is continuous at \( y_0 \). Moreover, since \( f \) is continuous at \( c \), \( h \circ f \) is continuous at \( c \) (Theorem 4.17).

Thus,

\[
\lim_{x \to c} (h \circ f)(x) = (h \circ f)(c) = h(y_0) = 0.
\]

We calculate

\[
(h \circ f)(x) = \frac{g(f(x)) - g(y_0)}{f(x) - y_0} - g'(y_0)
\]

if \( f(x) \neq y_0 \) and zero otherwise. Thus, for \( x \in I \) and \( x \neq c \)

\[
\frac{g(f(x)) - g(f(c))}{x - c} = \left( (h \circ f)(x) + g'(y_0) \right) \left( \frac{f(x) - f(c)}{x - c} \right).
\]

The result now follows from the algebra of limits. Indeed

\[
(g \circ f)'(c) = \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} = \lim_{x \to c} \left( (h \circ f)(x) + g'(y_0) \right) \cdot \lim_{x \to c} \left( \frac{f(x) - f(c)}{x - c} \right) = (0 + g'(y_0))f'(c) = g'(f(c))f'(c).
\]

\( \square \)

You might try to find a function which is differentiable but whose derivative is not continuous. Here is one.

**Example 5.7.** Set

\[
f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}
\]

Away from \( x = 0 \) the chain rule applies and \( f'(x) = 2x \sin(1/x) - \cos(1/x) \). At \( x = 0 \) we must apply the definition. We find

\[
f'(0) = \lim_{x \to 0} \frac{f(x) - 0}{x - 0} = 0.
\]

However, \( f' \) is not continuous at \( x = 0 \) since \( \lim_{x \to 0} f'(x) \) does not exist.
5.1. Mean-Value Theorems

In this section we establish some familiar results from elementary calculus.

**Definition 5.8.** Let \( f : I \to \mathbb{R} \) with \( I \) an open interval. A point, \( x_0 \in I \), is a relative maximum (relative minimum) of \( f \) if there exists \( \delta > 0 \) such that for all \( x \in I \) and \( |x - x_0| < \delta \), \( f(x_0) \geq f(x) \) (\( f(x_0) \leq f(x) \)).

**Theorem 5.9.** Let \( f : [a, b] \to \mathbb{R} \) and suppose \( f \) has either a relative maximum or a relative minimum at \( x_0 \in (a, b) \). If \( f \) is differentiable at \( x_0 \), then \( f'(x_0) = 0 \).

**Proof.** Suppose \( x_0 \) is relative maximum. Then, \( \delta > 0 \) exists so that, for any \( x \) satisfying \( x_0 - \delta < x < x_0 + \delta \), \( x \in [a, b] \) and \( f(x) \leq f(x_0) \). Since the derivative exists at \( x_0 \), we may compute it by taking any sequence \( \{x_n\}_{n \in \mathbb{N}} \subset [a, b]/\{x_0\} \) converging to \( x_0 \). We choose a sequence such that \( x_0 - \delta < x_n < x_0 \) for all \( n \in \mathbb{N} \). For this choice note that, since \( x_0 \) is a relative maximum \( f(x_n) \leq f(x_0) \),

\[
\frac{f(x_n) - f(x_0)}{x_n - x_0} \geq 0.
\]

This implies the limit is non negative and \( f'(x_0) \geq 0 \). Similarly we could choose \( \{y_n\}_{n \in \mathbb{N}} \subset [a, b]/\{x_0\} \) converging to \( x_0 \) with \( x_0 < y_n < x_0 + \delta \) for all \( n \in \mathbb{N} \). Then

\[
\frac{f(y_n) - f(x_0)}{y_n - x_0} \leq 0
\]

for \( n \in \mathbb{N} \). This implies the limit is non positive and \( f'(x_0) \leq 0 \). Thus, \( f'(x_0) = 0 \) as required. A similar argument holds for a relative minimum. \( \square \)

**Theorem 5.10.** (Rolle’s Theorem) Let \( f : [a, b] \to \mathbb{R} \) continuous on \([a, b]\) and differentiable on \((a, b)\). If \( f(a) = f(b) = 0 \), there exists \( c \in (a, b) \) such that \( f'(c) = 0 \).

**Proof.** If \( f(x) = 0 \) for all \( x \in [a, b] \), \( f'(x) = 0 \) and any point \( c \in (a, b) \) will suffice. Suppose \( f(x) \not= 0 \) for some \( x \in [a, b] \). By the Extreme-Value Theorem \( f \) assumes a minimum and maximum, \( x_M, x_m \) in \([a, b]\). Since \( f(a) = f(b) = 0 \), one of \( x_M, x_m \in (a, b) \), say \( x_M \). By the previous theorem \( f'(x_M) = 0 \). \( \square \)

The next theorem is a generalization of Rolle’s Theorem.

**Theorem 5.11.** (Mean-Value Theorem). Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\). Then, there exists \( c \in (a, b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

**Proof.** Set

\[
g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)
\]

Then \( g(a) = g(b) = 0 \), and by the algebra of continuous/differentiable functions \( g \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Rolle’s Theorem applies, and \( c \in (a, b) \) exists so that \( g'(c) = 0 \). But \( g'(c) = f'(c) - (f(b) - f(a))/(b - a) \). \( \square \)

The next theorem generalizes it even more.
5.2. Applications of the Mean-Value Theorem

**Theorem 5.12.** (Cauchy Mean-Value Theorem). Let \( f, g : [a, b] \rightarrow \mathbb{R} \) continuous on \([a, b]\) and differentiable on \((a, b)\). Then, there exists \( c \in (a, b) \) such that

\[
(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).
\]

**Remark 5.13.** If \( g(x) = x \) we recover the Mean-Value Theorem.

**Proof.** Set

\[
h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x),
\]

for \( x \in [a, b] \). Then, \( h(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Moreover, \( h(a) = h(b) \). The Mean-Value Theorem applies. Differentiating \( h \) gives the result. \( \square \)

**5.2. Applications of the Mean-Value Theorem**

The usefulness of the Mean-Value Theorem cannot be overstated. Interestingly it does not extend to higher spatial dimensions.

**Theorem 5.14.** Let \( f : [a, b] \rightarrow \mathbb{R} \) continuous on \([a, b]\) and differentiable on \((a, b)\).

(i) If \( f'(x) \neq 0 \) for all \( x \in (a, b) \), \( f \) is injective on \([a, b]\).

(ii) If \( f'(x) = 0 \) for all \( x \in (a, b) \), \( f \) is constant on \([a, b]\).

(iii) If \( f'(x) > 0 \) for all \( x \in (a, b) \), \( x, y \in [a, b], x < y \implies f(x) < f(y) \).

(iv) If \( f'(x) < 0 \) for all \( x \in (a, b) \), \( x, y \in [a, b], x < y \implies f(x) > f(y) \).

(v) If \( |f'(x)| \leq M \) for all \( x \in (a, b) \), and some \( M \) and we only require \( f \) to be differentiable on \((a, b)\), then \( f \) is uniformly continuous on \((a, b)\).

**Proof.** These follow immediately from the Mean-Value Theorem, and in particular the expression

\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]

We only check (v). We have, applying the Mean-Value Theorem, for any \( x, y \in (a, b) \)

\[
|f(x) - f(y)| \leq |f'(c_{xy})||x - y| \leq M|x - y|
\]

for some \( c_{xy} \in (a, b) \) and between \( x \) and \( y \). A function satisfying the inequality \( |f(x) - f(y)| \leq M|x - y| \) is called Lipschitz. It implies \( f \) is uniformly continuous on \((a, b)\) if we choose \( \delta = \epsilon/M \). \( \square \)

**Theorem 5.15.** Let \( f, g : [a, b] \rightarrow \mathbb{R} \) continuous on \([a, b]\), differentiable and \( f'(x) = g'(x) \) on \((a, b)\). Then, \( f(x) = g(x) + k \) for all \( x \in [a, b] \) and constant \( k \).

**Proof.** Set \( h(x) = f(x) - g(x) \). Then, \( h \) satisfies the conditions of Theorem 5.14, (ii). \( \square \)

The next theorem is unexpected since the derivative of a function need not be continuous (recall Example 5.7).

**Theorem 5.16.** (Darboux’s Theorem) Suppose \( f : [a, b] \rightarrow \mathbb{R} \) is differentiable on \([a, b]\). Then, \( f' \) enjoys the intermediate-value property.
Proof. Suppose that, for some differentiable function \( h : [a, b] \rightarrow \mathbb{R} \), \( h'(x) \neq 0 \) on \((a, b)\). Then, by Theorem 5.14, (i) \( h \) is injective. Moreover, by Exercise 4.24, \( h \) is monotone. If \( h \) is increasing
\[
\frac{h(x) - h(y)}{x - y} \geq 0.
\]
This implies \( h' \geq 0 \) on \([a, b]\). If \( h \) is decreasing, \( h' \leq 0 \) on \([a, b]\). Thus, \( h'(x) \neq 0 \) on \((a, b)\) implies either \( h'(x) \leq 0 \) or \( h'(x) \geq 0 \) on \([a, b]\) - \( h' \) cannot change signs.

To show \( f' \) has the intermediate-value property, let \( x, y \in [a, b] \), \( x < y \) and \( \lambda \) be any point between \( f'(x) \) and \( f'(y) \). Set \( h(z) = f(z) - \lambda z \). Then, \( h \) is differentiable on \([a, b]\) and \( h'(z) = f'(z) - \lambda \). One checks that under our assumptions about \( \lambda \), \( h'(x) \) and \( h'(y) \) have opposite signs. Thus, \( h' \) must be zero for some \( c \) between \( x \) and \( y \). That is, \( f'(c) = \lambda \) for some \( x < c < y \).

Example 5.17. There is no function on the interval \([-1, 1]\) whose derivative is
\[
h(x) = \begin{cases} 
-1 & x \leq 0, \\
1 & x > 0.
\end{cases}
\]

The next theorem is called L’Hospital’s Rule. It appeared in a calculus book written by L’Hospital: Analysis of the Infinitely Small to Understand Curved Lines, (1696). The result is thought to have been discovered by Johann Bernoulli.

Theorem 5.18. (L’Hospital’s Rule) Suppose \( f, g : [a, b] \rightarrow \mathbb{R} \) are continuous on \([a, b]\) and differentiable on \((a, b)\). Let \( x_0 \in [a, b] \), and

(i) \( g'(x) \neq 0 \) for all \( x \in (a, b) \),
(ii) \( f(x_0) = g(x_0) = 0 \),
(iii) \( f'/g' \) has a limit at \( x_0 \).

Then
\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.
\]

Proof. Let \( \{x_n\}_{n \in \mathbb{N}} \subset (a, b)/\{x_0\} \) be any sequence converging to \( x_0 \). By the Cauchy Mean-Value Theorem, Theorem 5.12, there exists a sequence \( \{c_n\}_{n \in \mathbb{N}} \) with \( c_n \) between \( x \) and \( x_0 \) for all \( n \in \mathbb{N} \) such that
\[
(f(x_n) - f(x_0))g'(c_n) = (g(x_n) - g(x_0))f'(c_n).
\]
Since \( g' \neq 0 \) on \((a, b)\), \( g \) is injective. Thus, \( g(x_n) \neq g(x_0) \) for all \( n \in \mathbb{N} \). Moreover, since \( f(x_0) = g(x_0) = 0 \), we have
\[
\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(x_0)}{g(x_n) - g(x_0)} = \frac{f'(c_n)}{g'(c_n)}.
\]
By the Squeeze Theorem, Theorem 3.24, \( \{c_n\}_{n \in \mathbb{N}} \) converges to \( x_0 \). The result follows.

Note L’Hospital’s rule says the existence of limit \( \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \) implies the limit \( \lim_{x \to x_0} \frac{f(x)}{g(x)} \) exists. The converse is not true.
Example 5.19. Set
\[ f(x) = \begin{cases} \sin \frac{1}{x} & x \leq 0, \\ 0 & x = 0. \end{cases} \]
and \( g(x) = \sqrt{x} \). Then, \( f(0) = g(0) = 0 \). Note
\[ \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\sqrt{x} \sin \frac{1}{x}}{x} = 0 \]
while
\[ \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \left( 2\sqrt{x} \sin \frac{1}{x} - \frac{2}{\sqrt{x}} \sin \frac{1}{x} \right), \]
does not have a limit.

Example 5.20. Prove
\[ \lim_{x \to 0^+} x \ln x = 0. \]

**Proof.** We can rewrite the limit as \( \lim_{x \to 0^+} \frac{\ln x}{1/x} = 0 \). The limit tends to \(-\infty/\infty\) which is not in the form discussed in Theorem 5.18. Nevertheless, the proof still works. Set \( f(x) = \ln x \) and \( g(x) = 1/x \). Notice \(-x = f'/g' \to 0\) as \( x \to 0^+ \). Thus, for any \( \epsilon > 0 \), there exists \( 0 < x_1 < 1 \) such that
\[ \frac{|f'(x)|}{g'(x)} < \epsilon \]
for all \( 0 < x < x_1 \). Applying the Cauchy Mean-Value Theorem, we find
\[ \left| \frac{f(x) - f(x_1)}{g(x) - g(x_1)} \right| = \left| \frac{f'(x_2)}{g'(x_2)} \right| < \epsilon \]
for some \( x_2 \) between \( x \) and \( x_1 \). However,
\[ \frac{f(x) - f(x_1)}{g(x) - g(x_1)} = \frac{f(x)}{g(x)} \left[ 1 - \frac{f(x_1)}{f'(x_2)} \right] = \frac{f(x)}{g(x)} h(x). \]
We see \( h(x) \to 1 \) as \( x \to 0^+ \). Thus, \( h(x) > 1/2 \) for \( x \) small enough, and
\[ \left| \frac{f(x)}{g(x)} \right| \leq \frac{f'(x_2)}{g'(x_2)} < \epsilon \]
for \( x \) small. Thus, given any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that
\[ |x \ln x - 0| = \left| \frac{f(x)}{g(x)} \right| < 2\epsilon, \]
and the limit is established. \[ \square \]

**Theorem 5.21.** (Taylor’s Theorem) Let \( f : [a, b] \to \mathbb{R} \). Suppose \( f^{(n)}(x) \) is continuous on \([a, b]\) and \( f^{(n+1)}(x) \) exists on \((a, b)\) for some \( n \in \mathbb{N} \). For any \( x, x_0 \in [a, b], c \) between \( x \) and \( x_0 \) exists such that
\[ f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots \]
\[ + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \]
\[ = P_n(x) + R_n(x). \]
The last term in the sum, \( R_n(x) \), is called the Lagrangian form of the remainder.

**Proof.** Note that, if \( n = 0 \), Taylor’s theorem is the mean-value theorem. For any \( n \in \mathbb{N} \) we define the number \( E \) to satisfy

\[
f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{E}{(n+1)!}(x-x_0)^{n+1}
\]

For any \( x, t \in [a, b] \) set

\[
\varphi(t) = f(x) - f(t) - f'(t)(x-t) - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - \frac{E}{(n+1)!}(x-t)^{n+1}.
\]

By the algebra of continuous/differentiable functions \( \varphi(t) \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Moreover, \( \varphi(x) = 0 = \varphi(x_0) \). By Rolle’s theorem there is \( c \) between \( x \) and \( x_0 \) such that \( \varphi'(c) = 0 \). We calculate

\[
\varphi'(t) = -f'(t) + f'(t) - \frac{f''(t)}{2!}(x-t)^2 + \frac{f''(t)}{2!}(x-t)^2 + \cdots - \frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{E}{n!}(x-t)^n.
\]

Since \( \varphi'(c) = 0 \), we find

\[
E = f^{(n+1)}(c).
\]

\[\Box\]

**Example 5.22.** Show

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots
\]

**Proof.** For the moment we assume \( f(x) = e^x \) has infinitely many derivatives on \( \mathbb{R} \) and \( f'(x) = e^x \). The number \( e \) is defined in Equation (5.2). Taylor’s Theorem applies and

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + e^c \frac{x^{n+1}}{(n+1)!}
\]

for some \( c \) between 0 and \( x \). That is, the remainder is

\[
R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}.
\]

For a fixed \( x \) consider the sequence \( \{a_n\}_{n \in \mathbb{N}} = \{|R_n|\}_{n \in \mathbb{N}} \). Problem 3.14 shows that any sequence \( \{a_n\}_{n \in \mathbb{N}} \) satisfying \( a_n > 0 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L \) with \( L < 1 \) converges to zero. Here

\[
\frac{a_{n+1}}{a_n} = \frac{|x|}{n+2},
\]

Thus, \( R_n \) goes to zero as \( n \to \infty \) for all \( x, c \). \[\Box\]
Example 5.23. In Example 1.3 we examined the Taylor expansion of
\[ f(x) = \begin{cases} e^{-\frac{x^2}{2}} & x \neq 0, \\ 0 & x = 0. \end{cases} \]
In Exercise 5.19, we find \( f^{(n)}(0) = 0 \) for all \( n \in \mathbb{N} \). Thus, \( P_n(x) = 0 \) and all of the information about the function is in the remainder \( R_n(x) \), and the Taylor expansion of \( f \) does not converge to \( f \).

A nice application of Taylor’s Theorem was produced by Charles Hermite (1873). We define the number \( e \) to be the unique number such that
\[
1 = \int_{1}^{e} \frac{1}{x} \, dx.
\]
The integral will be defined in the next chapter.

Theorem 5.24. The number \( e \) is irrational.

Proof. We know from Example 5.22 that \( e \) may be written
\[
e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{e^c}{(n+1)!}.
\]
Suppose \( e = p/q \) (reduced form) with \( p, q \in \mathbb{N} \). Fix \( n \geq \max\{q, e\} \). Then
\[
n!e = \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) n! + \frac{e^c}{n+1}.
\]
Now \( n!e = n!p/q \) and \( n \geq q \). Thus, \( n!e \in \mathbb{N} \), and therefore, so is \( \frac{e^c}{n+1} \). However \( n \geq e \), and so \( n+1 > e \) or \( e^c/(n+1) < 1 \), a contradiction. \( \square \)
Theorems so far. See if you remember the ideas of the proofs and the way the one theorem was used to prove another.
5.2. Applications of the Mean-Value Theorem

A Collection of Strange Functions

(a) The following function is differentiable for all $x \in \mathbb{R}$; however, its derivative is not continuous.

\[
g(x) = \begin{cases} 
  x^2 \sin(1/x) & x \neq 0, \\
  0 & x = 0.
\end{cases}
\]

(b) The function, $h(x)$, below is differentiable and continuous at exactly one point.

\[
h(x) = \begin{cases} 
  x^2 & x \in \mathbb{Q}, \\
  0 & x \not\in \mathbb{Q}.
\end{cases}
\]

(c) The following function is differentiable everywhere. In addition, $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, and the function has no Taylor series expansion.

\[
f(x) = \begin{cases} 
  \exp(-1/x^2) & x \neq 0, \\
  0 & x = 0.
\end{cases}
\]

(d) The next function is continuous on the irrationals but discontinuous on the rationals (it is also Riemann integrable on $[0,1]$).

\[
g(x) = \begin{cases} 
  1/q & x = p/q \in \mathbb{Q}, \text{ (reduced form)} \\
  0 & x \not\in \mathbb{Q}.
\end{cases}
\]

(e) The next function is not continuous anywhere (it is not Riemann integrable as we will see in the next chapter)

\[
f(x) = \begin{cases} 
  1 & x \in \mathbb{Q}, \\
  0 & x \not\in \mathbb{Q}.
\end{cases}
\]

(f) Consider the function on the real line given by

\[
f(x) = \begin{cases} 
  x & x \in \mathbb{Q}, \\
  1 - x & x \in \mathbb{Q}^c.
\end{cases}
\]

The function is continuous at $x = 1/2$ and discontinuous everywhere else.

(g) Finally, the function below is continuous everywhere, but differentiable nowhere:

\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos(3^n x).
\]

For a proof of (g) see [4], A Primer of Real Functions by Ralph P. Boas Jr., (1981), QA331.5 B63x.
5.3. Homework

Exercise 5.1. Using the definition find the derivative of \( f(x) = \sqrt{x} \) for \( x > 0 \).

Exercise 5.2. Let \( f \) be differentiable on an interval \( \mathbb{R} \). If \( f \) is monotone increasing on \( \mathbb{R} \), show \( f'(x) \geq 0 \) on \( \mathbb{R} \).

Exercise 5.3. Let \( f, g : [a, b] \to \mathbb{R} \), and suppose \( f, g \) are differentiable at \( x_0 \in (a, b) \). In addition, \( f(x_0) = g(x_0) \) and \( f(x) \leq g(x) \) for all \( x \in [a, b] \). Does \( f'(x_0) = g'(x_0) \)?

Exercise 5.4. Provide a more direct proof of Darboux’s Theorem (Theorem 5.16) by considering the maximum of the function \( g(t) = f(t) - yt \), where \( y \) satisfies \( f'(a) > y > f'(b) \). Hint: first show that, if a function satisfies \( g'(a) > 0 \) and \( g'(b) < 0 \), then the maximum must be in the interior.

Exercise 5.5. Let \( \epsilon > 0 \). Show there does not exist a differentiable function on \([0, \infty)\) with \( f'(0) = 0 \) and \( f'(x) \geq \epsilon \) for \( x > 0 \).

Exercise 5.6. Suppose \( f \) is differentiable on \( \mathbb{R} \) and that \( f \) has \( n \) distinct roots. Show \( f' \) has at least \( n - 1 \) distinct real roots. Show by example that \( f' \) can have more real roots than \( f \).

Exercise 5.7. Let \( f : (0, 1] \to \mathbb{R} \) be differentiable with \( |f'(x)| \leq 1 \) for \( x \in (0, 1] \). For each \( n \in \mathbb{N} \), set \( a_n = f(1/n) \). Show that the sequence \( \{a_n\}_{n \in \mathbb{N}} \) converges.

Exercise 5.8. Let \( f \) be continuous on \([0, \infty)\) and differentiable on \((0, \infty)\), and let \( f(0) = 0 \). If \( f' \) is monotone increasing on \((0, \infty)\), show that \( g(x) = f(x)/x \) is monotone increasing on \((0, \infty)\).

Exercise 5.9. Let \( f \) be continuous on \([a, b]\) and differentiable on \((a, b)\). Suppose \( \lim_{x \to a^+} f'(x) \) exists. Show \( f \) is differentiable at \( a \) and that \( f'(a) = \lim_{x \to a^+} f'(x) \).

Exercise 5.10. A function \( f : I \to \mathbb{R} \) is called uniformly Hölder continuous with exponent \( \theta \) if there exists a constant, \( c \), such that for all \( x, y \in I \), \( |f(x) - f(y)| \leq c|x - y|^{\theta} \) with \( 0 < \theta < 1 \). It is called uniformly Lipschitz if \( \theta = 1 \).

(a) Prove that Hölder and Lipschitz continuous functions are in fact continuous.
(b) What can you say about the differentiability of such functions?
(c) Give a characterization of the functions that are Hölder continuous with exponent \( \theta > 1 \).

Exercise 5.11. Recall a function \( f \) is said to have a fixed point if \( f(c) = c \).

(i) Suppose \( f \) is differentiable on \( \mathbb{R} \) and \( |f'(x)| < 1 \) for all \( x \in \mathbb{R} \). Show that \( f \) can have at most one fixed point.

(ii) Let \( f(x) = x + (1 + e^x)^{-1} \). Show \( f \) satisfies Part (i), but has no fixed points.

Exercise 5.12. Let \( P \) be any polynomial of degree two and let \([a, b]\) be any interval. Find the \( c \) guaranteed by the Mean-Value Theorem. Do you notice anything interesting about this point? Repeat for \( f(x) = 1/x \) with \( a > 0 \).

Exercise 5.13. Suppose \( \frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)} \) on some interval \( I \). Find, without using logarithms, a relationship between \( f \) and \( g \).
Exercise 5.14. Suppose $f, g$ are differentiable on $[a, b]$ an $f'(x) \geq g'(x)$ for all $x \in [a, b]$. Show $f(b) - f(a) \geq g(b) - g(a)$.

Exercise 5.15. Prove $\sqrt{1 + x} \leq 1 + x/2$ for all $x > 0$ using the Mean-Value Theorem.

Exercise 5.16. Prove $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$

Exercise 5.17. Show $(x - 1)/x < \ln x < x - 1$ for $x > 1$, and hence, $\ln(1 + x) < x$ for $x > 0$.

Exercise 5.18. Prove $e^x \geq 1 + x$ for all $x \geq 0$

Exercise 5.19. By using induction find a general form for the derivative of

$$f(x) = \begin{cases} e^{-x^2/2} & x \neq 0, \\ 0 & x = 0, \end{cases}$$

and hence, show $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Thus, the Taylor expansion for $f$ is zero.

Exercise 5.20. Show, for $x > 0$,

$$1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2} e^x.$$ 

Exercise 5.21. Suppose $f \in C^2([a, b])$ (f as two continuous derivatives on $[a, b]$), $f(a) = f(b) = 0$, and $f(x) > 0$ for all $x \in (a, b)$.

(a) Prove $\gamma \in (a, b)$ exists such that $f''(\gamma) < 0$.

(b) Prove there exists an interval $(\alpha, \beta) \subset [a, b]$ on which $f''(x) < 0$.

Exercise 5.22. Prove the second-derivative test: Suppose $f$ has two continuous derivatives on $[a, b]$, $f'(x_m) = 0$ for some $x_m \in (a, b)$, and $f''(x_m) > 0$. Prove $x_m$ is a local minimum.

Exercise 5.23. Let $C, S$ be two real-valued functions on the real line satisfying the following properties:

$$S(x + y) = S(x)C(y) + C(x)S(y), \quad C(x + y) = C(x)C(y) - S(x)S(y);$$

$$S(0) = 0, \quad S'(0) = 1, \quad C(0) = 1, \quad C'(0) = 0.$$

(a) Show $S'(x)$ exists for all $x \in \mathbb{R}$ and is $C(x)$.

(b) Show $C''(x)$ exists for all $x \in \mathbb{R}$ and is $-S(x)$.

(c) Use (a), (b), and Theorem 5.14 (ii) to show $S^2 + C^2 = 1$.

(d) show $S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

(e) Show that $S(x) = \sin(x)$.

Exercise 5.24. (a) Show that $|x| < e^{x^2}$ for all real $x$.

(b) Show that $f(x) = e^{-x^2}$ is uniformly continuous on $\mathbb{R}$. (Hint: use the MVT and Part (a)).