Chapter 2

Structure of \( \mathbb{R} \)

We will re-assemble calculus by first making assumptions about the real numbers. All subsequent results will be rigorously derived from these assumptions. Most of the assumptions will seem familiar, some will not. It would be possible to start with the natural numbers, an obvious more aesthetic starting point, but we shall not do so.

2.1. Algebraic and Order Properties of \( \mathbb{R} \)

We assume the set of real numbers, \( \mathbb{R} \), has two binary operations, denoted + and \( \cdot \) (i.e. a function whose domain is \( \mathbb{R} \times \mathbb{R} \) and range in \( \mathbb{R} \)). The operations are called addition and multiplication respectively. Moreover, they satisfy the following properties:

\( (A1) \)
\[
\begin{align*}
  a + b &= b + a \\
  a \cdot b &= b \cdot a
\end{align*}
\]
(commutative laws)

\( (A2) \)
\[
\begin{align*}
  a + (b + c) &= (a + b) + c \\
  a \cdot (b \cdot c) &= (a \cdot b) \cdot c
\end{align*}
\]
(associative laws)

\( (A3) \)
\[
  a \cdot (b + c) = a \cdot b + a \cdot c
\]
(distributive law)

\( (A4) \) There exists unique, distinct real numbers 0 and 1 such that, for all \( a \in \mathbb{R} \),
\[
\begin{align*}
  a + 0 &= a \\
  a \cdot 1 &= a
\end{align*}
\]
(identity elements)

\( (A5) \) For each \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \), \( b \neq 0 \), there exists \( -a \in \mathbb{R} \) and \( b^{-1} := 1/b \in \mathbb{R} \) such that
\[
\begin{align*}
  a + (-a) &= 0 \\
  b \cdot b^{-1} &= 1
\end{align*}
\]
(inverse elements)
The above axioms show that \((\mathbb{R}, +, \cdot)\) is a “field” in the sense of abstract algebra. We note that \((\mathbb{Q}, +, \cdot)\) is a field, where \(\mathbb{Q}\) is the set of rational numbers. This is so since the sum or product of two fractions is again a fraction. That is, the rational numbers are closed under addition and multiplication.

**Order Axioms for \(\mathbb{R}\)**

We assume the existence of an order property on \(\mathbb{R}\), denoted by \(<\), with the following properties:

(O1) For all \(a, b \in \mathbb{R}\), exactly one of the following holds (trichotomy): \(a = b\), \(a < b\), or \(b < a\).

(O2) For all \(a, b, c \in \mathbb{R}\), if \(a < b\), then \(a + c < b + c\).

(O3) For all \(a, b, c \in \mathbb{R}\), if \(a < b\) and \(0 < c\), then \(ac < bc\).

(O4) For all \(a, b, c \in \mathbb{R}\), if \(a < b\) and \(b < c\), then \(a < c\) (transitivity).

**2.2. Absolute Value**

We need a way of measuring the size of real numbers. In one dimension we do this with the absolute-value function. It is defined by

\[
|x| := \begin{cases} 
  x & x \geq 0 \\
  -x & x < 0
\end{cases}.
\]

**Theorem 2.1.** Suppose \(a, b \in \mathbb{R}\). Then

(i) \(|a| = 0\) if and only if \(a = 0\).

(ii) \(|-a| = |a|\).

(iii) \(|ab| = |a||b|\).

(iv) Suppose \(M \geq 0\). Then \(|a| \leq M\) if and only if \(-M \leq a \leq M\).

(v) (Triangle Inequality)

\[|a \pm b| \leq |a| + |b|\]

\[||a| - |b|| \leq |a \pm b|\]

**Proof.** To prove \((i)\) suppose \(|a| = 0\). By definition \(\pm a = 0\). That is \(a = 0\). The converse follows in the same way. To prove \((ii)\) suppose \(a > 0\). Then \(-a < 0\), and by definition \(|-a| = -( -a) = a = |a|\). If \(-a \geq 0\), then the definition implies \(|-a| = -a = |a|\). To prove \((iii)\) note that \(ab = \pm|a||b|\) depending on the sign of \(a\) and \(b\). Part \((ii)\) gives the result. Part \((iv)\) is similar. Suppose \(|a| \leq M\). If \(a \geq 0\), \(-M \leq 0 \leq a = |a| \leq M\). If \(a < 0\), \(-M \leq -|a| = a < 0 \leq M\), and \(-M \leq a \leq M\). Conversely, suppose \(-M \leq a \leq M\). Then \(-M \leq a \) or \(-a \leq M\). Since \(a \leq M\) as well, \(|a| \leq M\).

Finally to prove the triangle inequality, we start with the obvious inequalities \(|a| \leq |a|\), \(|b| \leq |b|\) and apply \((iv)\). We find

\[-|a| \leq a \leq |a|\]

\[-|b| \leq b \leq |b|\]
Adding the last two equations shows \(-(|a| + |b|) \leq a + b \leq |a| + |b|\). Part (iv) applies again and the first inequality in (v) follows. For the other inequality we compute, using the first inequality (the triangle inequality) in (v),

\[ |a| = |(a - b) + b| \leq |a - b| + |b|. \]

That is,

\[ |a| - |b| \leq |a - b|. \]

Reversing the roles of \(a\) and \(b\) we find

\[ |b| - |a| \leq |a - b|. \]

This implies \(|a| - |b| \geq -|a - b|\), and (iv) gives the second inequality in (v). \(\square\)

**Example 2.2.** Estimate the size of the function \(f(x) = x^2 - 1\) on the interval \(0 < x < 2\).

**Solution.** Of course we could easily figure this out by putting appropriate numbers into the function, but the idea is to gain experience using the triangle inequalities. Indeed,

\[ |x^2 - 1| = |(x - 1)(x + 1)| \leq |x - 1||x + 1|. \]

We may write the domain of the function as \(|x - 1| < 1\). Using the second of the triangle inequalities

\[ 1 \geq |x - 1| \geq |x| - 1 \geq |x| - 1. \]

It follows \(|x| \leq 2\). Thus

\[ |x^2 - 1| \leq |x - 1||x + 1| \leq 1 \cdot (|x| + 1) \leq 3. \]

on the interval of interest.

### 2.3. Supremum and Infimum

We need two more assumptions about the structure of the real numbers. We first clean up a popular misconception - the minimum and maximum of a set may not exist!

**Example 2.3.** Consider \(S = \{ \frac{1}{x} \mid 1 \leq x < \infty \}\). What is the smallest element in \(S\)?

We better be clear what we mean by maximum and minimum.

**Definition 2.4.** (Maximum) Let \(S \subset \mathbb{R}\) be non empty. The maximum of \(S\), \(M\), written \(M = \max S\), has the following properties:

(i) \(x \leq M\) for all \(x \in S\).

(ii) \(M \in S\).

A similar definition applies for the minimum.

Since the minimum and maximum may not exist, we seek a way to generalize the notion of smallest and largest in such a way that they agree with minimum and maximum when they exist, and in such a way that the generalization always exists. Here is how we do it.
Definition 2.5. Let \( S \subset \mathbb{R} \) be non empty.

(i) We say \( S \) is bounded above if \( M \in \mathbb{R} \) exists such that \( x \leq M \) for all \( x \in S \). Such \( M \) are called an upper bound of \( S \).

(ii) We say \( S \) is bounded below if \( m \in \mathbb{R} \) exists such that \( x \geq m \) for all \( x \in S \). Such \( m \) are called a lower bound of \( S \).

(iii) We say \( S \) is bounded if it bounded above and below.

Definition 2.6. (Supremum) Let \( S \subset \mathbb{R} \) be non empty. The supremum of \( S \), \( \alpha \), and written \( \alpha = \sup S \) (sometimes denoted \( \alpha = \text{lub} S \) for least upper bound), has the following properties.

(i) The element \( \alpha \) is an upper bound of \( S \). That is, \( x \leq \alpha \) for all \( x \in S \).

(ii) The element \( \alpha \) is the least upper bound of \( S \). That is, \( \alpha \leq M \) for any other upper bound, \( M \), of \( S \).

To find a maximum we typically look in the set for the largest value. Note to find a supremum we look outside the set for the smallest value bounding the set. A similar definition holds for infimum.

Definition 2.7. (Infimum) Let \( S \subset \mathbb{R} \) be non empty. The infimum of \( S \), \( \beta \), and written \( \beta = \inf S \) (sometimes denoted \( \beta = \text{glb} S \) for greatest lower bound), has the following properties.

(i) The element \( \beta \) is a lower bound of \( S \). That is, \( x \geq \beta \) for all \( x \in S \).

(ii) The element \( \beta \) is the greatest lower bound of \( S \). That is, \( \beta \geq m \) for any other lower bound, \( m \), of \( S \).

We do not quite have the techniques to prove a number is or is not the supremum of a set. For now, we make the following claims.

Example 2.8. Consider \( S = \{ x \mid 0 < x < 1 \} \). Then \( \sup S = 1 \) and \( \inf S = 0 \).

Example 2.9. Consider \( S = \{ x \mid x = 2 - 1/n, n \in \mathbb{N} \} \). Then \( \sup S = 2 \) and \( \inf S = 1 \).

No harm has been done - if the maximum exists, then the supremum agrees with the maximum:

Theorem 2.10. Suppose \( E \subset \mathbb{R} \) has a maximum and a supremum. Then \( \sup E = \max E \).

Proof. We need only check that \( M = \max E \) satisfies the conditions of Definition 2.6. By (i) in the definition of maximum, \( x \leq M \) for all \( x \in E \). This is (i) in the definition of the supremum. Thus \( M \) is an upper bound of \( E \). To show \( M \) is the least upper bound suppose \( M_2 \) is any other upper bound of \( E \). Then \( x \leq M_2 \) for all \( x \in E \). Since \( M \in E \), we have \( M \leq M_2 \). This shows \( M \) is the smallest upper bound and \( M = \sup E \).

\( \square \)
Theorem 2.11. The supremum of a set $E \subset \mathbb{R}$, if it exists, is unique.

Proof. Suppose $\alpha_1$ and $\alpha_2$ are supremums. This means they are both upper bounds of $E$. Since $\alpha_1$ is a least upper bound, $\alpha_1 \leq \alpha_2$. Similarly, $\alpha_2 \leq \alpha_1$, and so $\alpha_1 = \alpha_2$. \hfill \Box

There is an alternate way of looking at Definition 2.6 which is sometimes more convenient to use. The following theorem is equivalent to our definition of supremum.

Theorem 2.12. Let $S \subset \mathbb{R}$ be non empty. Then $\alpha = \sup S$ if and only if it has the following properties.

(i) The element $\alpha$ is an upper bound of $S$.

(ii) Any $y < \alpha$ is not an upper bound of $S$. That is, if $y < \alpha$, there exists $w \in S$ such that $y < w$.

Proof. Suppose $\alpha$ satisfies (i) and (ii) in the theorem. Property (i) is the same as (i) in the Definition 2.6. If $\alpha_1 < \alpha$, (ii) shows $\alpha_1$ is not an upper bound of $S$. That is, all other upper bounds are bigger or equal to $\alpha$. That is, $\alpha$ is the least upper bound.

Conversely, if $y < \sup S$, $y$ is not an upper bound of $S$. Thus $w \in S$ exists such that $y < w$. This is (ii). \hfill \Box

Example 2.13. Set $E = \{x \mid 0 < x < 1\}$. Prove $\sup E = 1$.

Proof. We apply Theorem 2.12. By the way $E$ is constructed there are no elements in $E$ larger than 1, and (i) follows. Suppose $0 < y < 1$. Set $w = \frac{y+1}{2}$. Then $y = \frac{w+y}{2} < \frac{w+1}{2} = w < \frac{y+1}{2} = 1$. Thus $y < w \in E$ and (ii) follows. Similarly, if $y \leq 0$, set $w = 1/2$, and again (ii) follows.

Alternatively we could have applied the definition of the supremum. To do this we first prove the following result: suppose $a \in \mathbb{R}$ and $a \leq \epsilon$ for all $\epsilon > 0$. Then $a \leq 0$. To see this we prove the contrapositive. Suppose $a > 0$. Set $\epsilon = a/2$. Then $a > \epsilon$. That is, $\epsilon > 0$ exists so that $a > \epsilon$. This is the contrapositive. (Compare this with Problem (2.3).)

We have already established 1 is an upper bound. Suppose $\alpha'$ is any other upper bound. We need to show $1 \leq \alpha'$. By the way $E$ is constructed, $1 - \epsilon \leq \alpha'$ for all $\epsilon > 0$. That is, $1 - \alpha' \leq \epsilon$ for all $\epsilon > 0$. This shows $1 - \alpha' \leq 0$ via the previous paragraph. That is, $1 \leq \alpha'$ and 1 is the least upper bound of $E$. \hfill \Box

Example 2.14. For a more abstract example suppose $A$ and $B$ are two collections of real numbers and that both $A$, $B$ have a supremum. Set $\alpha_A = \sup A$ and $\alpha_B = \sup B$. Consider the set $E = \{x + y \mid x \in A, y \in B\}$. Prove $\sup E = \alpha := \alpha_A + \alpha_B$.

Proof. We first need to show $\alpha$ is an upper bound of $E$. Suppose $z \in E$. Then $x \in A$ and $y \in B$ exist so that $z = x + y$. By the properties of $\alpha_A$ and $\alpha_B$, $x \leq \alpha_A$ and $y \leq \alpha_B$. It follows $z = x + y \leq \alpha_A + \alpha_B$, and so $\alpha$ is an upper bound of $E$.

We show $\alpha$ is the least upper bound two ways - by applying the definition and applying Theorem 2.12. To apply the definition suppose $\alpha'$ is another upper bound
of $E$. We show $\alpha \leq \alpha'$. (This implies $\alpha$ is the least upper bound.) Since $\alpha'$ is an upper bound of $E$, $x + y \leq \alpha'$ for all $x \in A$ and $y \in B$. Thus $x \leq \alpha' - y$ for all $x \in A$. This means $\alpha' - y$ is an upper bound of $A$. Thus $\alpha_A \leq \alpha' - y$ since $\alpha_A$ is the least upper bound of $A$. It follows $y \leq \alpha' - \alpha_A$ for all $y \in B$. As before this implies $\alpha_B \leq \alpha' - \alpha_A$. That is, $\alpha_A + \alpha_B \leq \alpha'$ and $\alpha$ is the least upper bound of $E$.

Alternatively we could apply Theorem 2.12. We have already established $\alpha$ is an upper bound of $E$. Next we apply $(ii)$ in the theorem. Suppose $z < \alpha$. We need to show $z$ is not an upper bound. Set $x = \alpha_A - (\alpha_A + \alpha_B - z)/2$ and $y = \alpha_B - (\alpha_A + \alpha_B - z)/2$. Then $x + y = z$. Moreover, by construction $x < \alpha_A$ and $y < \alpha_B$. Theorem 2.12 $(ii)$ applies to the sets $A$ and $B$. Thus $w_x \in A$ exists so that $x < w_x \leq \alpha_A$ and $w_y \in B$ exists so that $y < w_y \leq \alpha_B$. This implies $w_x + w_y \leq \alpha_A + \alpha_B$. If we set $w = w_x + w_y$, then $w \in E$ and $z < w \leq \alpha$. This shows any $z < \alpha$ is not an upper bound of $E$. Theorem 2.12 $(ii)$ implies $\alpha$ is the least upper bound of $E$. \hfill \Box

We are ready for our last assumption about the real numbers. While a maximum of a bounded set may not exist, the supremum of a bounded set always exits. Indeed, we assume

**Completeness Axiom.** Every non empty bounded above subset of the real numbers has a supremum.

If we had more time, we could construct the real numbers starting with the natural numbers with the completeness axiom part of the construction. Moreover, we could have made a similar assumption about the infimum instead. Not surprisingly this is not necessary since we can get the result from the completeness axiom.

**Theorem 2.15. (Infimum Property)** Every non empty bounded below subset of the real numbers has an infimum.

**Proof.** Suppose $E$ is a non empty set bounded below. Set $E_1 = -E = \{-x \mid x \in E\}$. We expect that $\beta = \inf E = -\sup E_1 = -\alpha$. We must first establish the existence of $\alpha$. To do so we show $E_1$ is bounded above. Since $E$ is bounded from below, there exists a $m$ such that $x \geq m$ for all $x \in E$. That is, $-x \leq -m$, and all elements in $E_1$ are bounded by $-m$. Thus $E_1$ is a non empty bounded above set. By the completeness axiom it has a supremum, say $\alpha$.

Now we claim $\beta = -\alpha$ is an infimum for $E$. We check $\beta$ is a lower bound and that it is the greatest lower bound:

(i) Since $\alpha$ is an upper bound for $E_1$, $-x \leq \alpha$ for all $x \in E$. That is, $x \geq -\alpha = \beta$ for all $x \in E$, and $\beta$ is a lower bound for $E$.

(ii) Suppose $\beta < y$. That is, $-\alpha < y$ or $-y < \alpha$. By Theorem 2.12 there exists a $w \in E_1$ with $-y < w$ or $y > -w$. But $-w \in E$, and thus $y$ cannot be a lower bound. Thus $\beta$ is the greatest lower bound. \hfill \Box
2.4. Applications of the Completeness Axiom

Theorem 2.16. (Archimedean Property) The natural numbers are unbounded.

Proof. Suppose to the contrary the natural numbers are bounded. Then a $M$ exists so that $n \leq M$ for all $n \in \mathbb{N}$. By the completeness axiom an $\alpha$ exists so that $\alpha = \sup \mathbb{N}$. Now $\alpha - 1$ is no longer an upper bound of $\mathbb{N}$, so (by Theorem 2.12) an $n_\alpha \in \mathbb{N}$ exists with $\alpha - 1 < n_\alpha$ or $\alpha < n_\alpha + 1$. But $n_\alpha + 1 \in \mathbb{N}$, contradicting $\alpha$ being an upper bound. \qed

Looking in other books you will find different versions of the Archimedean Property. In an effort to bring the people of the world together we have

Theorem 2.17. (Archimedean Property) The following statements are equivalent.

(i) The natural numbers are unbounded.
(ii) For all $x \in \mathbb{R}$, there exists $n_x \in \mathbb{N}$ such that $x < n_x$.
(iii) For all $x > 0$, there exists $n_x \in \mathbb{N}$ such that $0 < 1/n_x < x$.

Proof. We show (i) implies (ii). Given any $x \in \mathbb{R}$ there is a $n_x \in \mathbb{N}$ such that $x < n_x$. This is (ii). Next, we show (ii) implies (iii). If $x > 0$, there is a $n_x \in \mathbb{N}$ such that $0 < 1/x < n_x$. Cross multiplying gives (iii). Finally, we show not (i) implies not (iii). If the natural numbers are bounded, there exists $M$ such that $n \leq M$ for all $n \in \mathbb{N}$. That is, $\frac{1}{M} \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Choose $x = 1/M$. Then there is no $n_x$ such that $\frac{1}{n_x} < \frac{1}{M}$. \qed

We note that as a consequence of the Archimedean Property there are infinitely many natural numbers, and hence infinitely many rational numbers, and infinitely many irrational numbers ($\sqrt{2}/n$ is irrational for $n \in \mathbb{N}$).

We have one final assumption about the natural numbers

Well-Ordering Property. Every nonempty subset of the natural numbers has a smallest element.

Theorem 2.18. (Density of Rationals) For any $x, y \in \mathbb{R}$ with $x < y$ there exists $r \in \mathbb{Q}$ such that $x < r < y$.

Proof. We first assume $x > 0$. By the Archimedean Property there exist $k, m \in \mathbb{N}$ such that

$$0 < \frac{1}{m} < y - x$$

and $k > mx$ or $k/m > x$. Let $n$ be the smallest such $k$ (which exists by the Well-Ordering Property). Thus

$$\frac{n}{m} > x \quad \text{and} \quad \frac{n - 1}{m} \leq x < \frac{n}{m}.$$ 

We claim $n/m < y$ for otherwise

$$\frac{n - 1}{m} \leq x < y \leq \frac{n}{m}.$$
This would imply
\[ y - x \leq \frac{n}{m} - x \leq \frac{n}{m} - \frac{n - 1}{m} = \frac{1}{m} \]
contrary to our choice for \( m \). Thus
\[ x < \frac{n}{m} < y \]
as required. The case \( x = 0 \) follows from Theorem 2.17 (iii). If \( x < y \), choose \( r = 0 \). If \( x < -y \), apply the above result to \( 0 < -y < -x \) to get \( 0 < -y < r < -x \) for some \( r \in \mathbb{Q} \). Then \( x < -r < y < 0 \) as required.

Corollary 2.19. (Density of Irrational Numbers) Let \( x < y \). There exists \( \xi \in \mathbb{Q}^c \) such that \( x < \xi < y \).

Proof. By the previous theorem there exists \( r \in \mathbb{Q} \) such that \( x < \frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}} \). That is, \( x < r\sqrt{2} < y \).

Corollary 2.20. Irrational numbers can be approximated by rational numbers.

Proof. Let \( \xi \in \mathbb{Q}^c \). For each \( n \in \mathbb{N} \) there exists \( r_n \in \mathbb{Q} \) such that
\[ \xi < r_n < \xi + \frac{1}{n} \]

Theorem 2.21. Every positive real number has a decimal expansion.

Proof. By the Well-Ordering Property, there is a \( d_0 \in \mathbb{N} \) such that
\[ d_0 \leq x < d_0 + 1. \]
Let \( d_1 \) be the largest integer such that
\[ d_0 + \frac{d_1}{10} \leq x. \]
Note that \( 0 \leq d_1 \leq 9 \). Let \( d_2 \) be the largest integer such that
\[ d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} \leq x, \]
and again note that \( 0 \leq d_2 \leq 9 \). Suppose for some integer \( n \geq 2 \), \( d_0, d_2, \ldots, d_{n-1} \) have been chosen this way. Let \( d_n \) be such that
\[ d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_{n-1}}{10^{n-1}} + \frac{d_n}{10^n} \leq x, \]
and note that \( 0 \leq d_n \leq 9 \). By induction this defines \( d_n \) for all positive integers. Let
\[ E = \left\{ d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} \mid n \in \mathbb{Z}^+ \right\}. \]
The set \( E \) is nonempty and bounded above by \( x \). By the Completeness Axiom, \( \alpha = \sup(E) \) exists. Since \( x \) is an upper bound of \( E \), \( \alpha \leq x \). We wish to show \( x = \alpha \).

Suppose \( \alpha < x \). Then by the Archimedean Property there exists a \( p \in \mathbb{N} \) such that \( 1/p < x - \alpha \). Using the definition of \( d_p \), we find
\[ x < d_0 + \frac{d_1}{10} + \cdots + \frac{d_p}{10^p} + \frac{1}{10^p} < \alpha + \frac{1}{p} < x, \]
2.4. Applications of the Completeness Axiom

a contradiction. Therefore $x = \alpha$. That is,

$$d_0.d_1d_2 \ldots = d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \ldots$$

is a decimal expansion of $x$. □

**Theorem 2.22.** If $p$ is any positive real number, there is a positive real number $x$ such that $x^2 = p$.

**Proof.** First we will prove the theorem for $p \geq 1$. Define

$$E := \{ z \in \mathbb{R} \mid 0 < z^2 \leq p \}.$$  

Now $E$ is nonempty since $1 \in E$, and $E$ is bounded from above. To see the latter, suppose $z > p$, then $z^2 > p^2 \geq p$ (since $p \geq 1$). Hence $z$ is not in $E$. Therefore, $p$ is an upper bound for $E$.

By the completeness axiom the supremum of $E$ exists. Let $x = \sup E$. We will show that $x^2 = p$. Suppose $x^2 < p$. Let $\delta < 1$ be the smaller of $1$ and $(p - x^2)/(2x + 1)$. Then

$$(x + \delta)^2 = x^2 + 2\delta x + \delta^2 \leq x^2 + 2\delta x + \delta = x^2 + \delta(2x + 1) \leq x^2 + p - x^2 = p.$$  

Therefore, $x + \delta$ belongs to $E$, and $x < x + \delta$ contrary to $x$ being an upper bound for $E$. Thus, $x^2 \geq p$.

Suppose now that $x^2 > p$. Let $\delta = (x^2 - p)/(2x)$. Then

$$(x - \delta)^2 = x^2 - 2\delta x + \delta^2 \geq x^2 - 2\delta x = x^2 - (x^2 - p) = p.$$  

This means $x - \delta$ is less than $x$ but serves as an upper bound for $E$, contrary to $x$ being the least upper bound. We conclude that $x^2 = p$.

Finally, if $0 < p < 1$, then $(1/p) > 1$, and so the proof applies to $1/p$. Thus, there is a positive real number $x$ such that $x^2 = 1/p$, and so $(1/x)^2 = p$. □
2.5. Homework

Exercise 2.1. Suppose \(a \leq x \leq b\), and \(a \leq y \leq b\). Show \(|x - y| \leq b - a\) and interpret the result geometrically.

Exercise 2.2. Show \(|a + b| = |a| + |b|\) if and only if \(ab \geq 0\). Try to avoid the use of cases.

Exercise 2.3. Let \(a \in \mathbb{R}\) and suppose \(|a| < \epsilon\) for all \(\epsilon > 0\). Prove \(a = 0\).

Exercise 2.4. Prove \(|a| = \sqrt{a^2}\).

Exercise 2.5. (a) Prove \(|x| \leq 1\) implies \(|x^2 - x - 2| \leq 3|x + 1|\).

(b) Prove that \(|x - 1| \leq 1\) implies \(|x^3 + x - 2| \leq 8|x - 1|\).

Exercise 2.6. Prove that the set \(E = \{x \mid 0 < x < 1\}\) has no maximum.

Exercise 2.7. Give an example of a set of rational numbers which is bounded but does not have a rational supremum.

Exercise 2.8. Give an example of a set of irrational numbers that has a rational supremum.

Exercise 2.9. Let \(A\) and \(B\) be nonempty bounded subsets of the real numbers. Prove \(\inf(A + B) = \inf A + \inf B\).

Exercise 2.10. Let \(A\) and \(B\) be nonempty bounded subsets of the real numbers. We define \(AB = \{xy \mid x \in A, y \in B\}\). Is it the case that \(\sup(AB) = \sup A \sup B\)?

Exercise 2.11. Let \(A\) and \(B\) be nonempty bounded subsets of the real numbers with \(A \subseteq B\). Show that

\[
\inf B \leq \inf A \leq \sup A \leq \sup B.
\]

Exercise 2.12. Let \(A\) and \(B\) be nonempty bounded subsets of the real numbers. Prove

\[
\sup(A \cup B) = \max\{\sup A, \sup B\}
\]

and

\[
\inf(A \cup B) = \min\{\inf A, \inf B\}
\]

Exercise 2.13. Show that a nonempty finite subset of the real numbers has both a minimum and a maximum (use induction and the previous problem).

Exercise 2.14. Suppose a nonempty subset of the real numbers is bounded above and the supremum is not in the subset. Show the subset must have an infinite number of elements.

Exercise 2.15. Let \(f\) and \(g\) be bounded real-valued functions on a nonempty set \(D\). Show that

\[
\sup_{x \in D} (f(x) + g(x)) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x).
\]

Exercise 2.16. Let \(f\) be bounded real-valued functions on a nonempty set \(D\). Show that

\[
\sup_{x \in D} (f^2(x)) = \left(\sup_{x \in D} |f(x)|\right)^2.
\]