Devastating Examples

Students of calculus develop an intuition for the subject typically based on geometric considerations. This intuition works adequately for problems in physics and engineering. However, when the subject is examined more rigorously, exceptions and counter examples emerge and the engineering view of calculus is seen as a small portion of a vast and elegant subject. In this chapter we provide some examples to illustrate calculus may not be as intuitive as a freshman course would indicate.

Example 1.1. We can find trouble pretty quickly. Consider for example the superlatives largest, smallest, tallest, shortest, etc... which are certainly ubiquitous in physics and engineering language. However consider the set

\[ E = \{ x | 0 < x < 1 \}. \]

What is the largest (maximum) and smallest (minimum) elements in \( E \)? One of our first tasks will be to develop a language to replace maximum and minimum when they do not exist.

Example 1.2. A professional engineer at an Ivy League school once made the following statement.

Theorem. Suppose \( f \) is a function defined on \( 0 \leq x < \infty \) and

\[ \int_0^\infty |f(x)| \, dx < \infty. \]

Then \( \lim_{x \to \infty} |f(x)| = 0. \)

After stating the claim mathematicians in the audience growled and the engineer knew he was in trouble. The statement is false but ostensibly reasonable. Indeed, the statement looks a bit like the \( n^{th} \) term test for infinite series. Recall that an infinite series diverges if the \( n^{th} \) term does not converge to zero. Thus, a “proof” might go something like this.
“Proof.” Suppose \( \lim_{x \to \infty} |f(x)| \neq 0 \). That is, \( \lim_{x \to \infty} |f(x)| = a > 0 \). Then for \( x \) large enough, say \( x \geq M \), \( |f(x)| \geq a/2 \), and

\[
\int_0^\infty |f(x)| \, dx \geq \int_M^\infty |f(x)| \, dx \geq \int_M^\infty \frac{a}{2} \, dx = \infty.
\]

This proves the contrapositive.

The theorem is not correct as the following example shows. Consider the function

\[
f(x) = \begin{cases} 
1 & \text{if } n \leq x \leq n + 1/2^n, \ n \in \mathbb{N}, \\
0 & \text{otherwise}.
\end{cases}
\]

Then

\[
\int_0^\infty |f(x)| \, dx = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,
\]

but \( \lim_{x \to \infty} |f(x)| \) does not exist. Indeed, that is the error in the above proof. The negation of \( \lim_{x \to \infty} |f(x)| = 0 \) is not \( \lim_{x \to \infty} |f(x)| = a > 0 \).

Of course the engineer would ask what condition will make the theorem correct. The mathematician should have an answer (see Homework 6.35).

**Example 1.3.** An engineer would certainly expect any function to have a Taylor expansion. That is,

\[
f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots
\]

However, consider the function

\[
f(x) = \begin{cases} 
e^{-\frac{1}{x^2}} & x \neq 0, \\
0 & x = 0.
\end{cases}
\]
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In Homework problem 5.19, we will show that \( f^{(n)}(0) = 0 \) - any derivative of \( f \) at the origin is zero. Figure 2 shows a graph of \( f \). It is quite flat near the origin. Indeed, its Taylor series is zero!

![Figure 2. Graphs of \( f(x) = e^{-1/x^2} \).](image)

The function is clearly not zero, so it does not equal its Taylor expansion. The function \( f \) in this case is a smooth nice function, yet it is badly misbehaved from an engineer’s perspective.

Sequences of functions are common in physics and engineering. They frequently originate from an approximation of some quantity. For example, consider the differential equation

(1.1) \[ y'' - y = 0. \]

That is, we want to know all functions whose second derivative minus itself is zero. An engineer might guess the solution has the form of a Taylor series,

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n. \]

Assuming we may take the derivative of the sum by differentiating each term,

(1.2) \[ y''(x) = \sum_{n=2}^{\infty} n(n - 1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n + 2)(n + 1) a_{n+2} x^n. \]

Subtracting the two and equating the difference to zero, we find

\[ \sum_{n=0}^{\infty} \left( (n + 2)(n + 1) a_{n+2} - a_n \right) x^n = 0. \]

Since the polynomial on the right is zero, the coefficients of the polynomial on the left should be zero. That is,

\[ a_{n+2} = \frac{a_n}{(n + 1)(n + 2)}. \]
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If we were desperate to find any solution, we might set $a_0 = a_1 = 1$ and we would find $a_2 = 1/2$, $a_3 = 1/(3 \cdot 2)$, $a_4 = 1/(4 \cdot 3 \cdot 2)$. In general, $a_n = 1/n!$. Then we hope that the sequence of functions given by

$$y_n(x) = \sum_{m=0}^{n} \frac{x^m}{m!}$$

converges to the solution of (1.1). In order to make this procedure rigorous (which we will do!), we need to justify the above steps. The next example shows that interchanging derivatives (or integrals) with limits, as we did in calculating (1.2), is not so simple.

Example 1.4. Consider the sequence of functions on $\mathbb{R}$ given by $f_n(x) = \frac{1}{\sqrt{n}} \sin nx$. In (1.2) we tacitly assumed that

$$\lim_{n \to \infty} (f_n(x))' = \lim_{n \to \infty} f_n'(x)$$

Here, the functions converge, for each $x \in [0, 1]$, to zero. So we say $\lim_{x \to \infty} f_n(x) = 0$. Thus $\lim_{n \to \infty} f_n(x)' = 0$. But $\lim_{n \to \infty} f_n'(x) = \lim_{n \to \infty} \sqrt{n} \cos(nx)$ which has no limit.

Example 1.5. Consider the sequence of functions given by $f_n(x) = nxe^{-nx^2}$ on $[0, 1]$.

As in the previous example, $\lim_{n \to \infty} f_n(x) = 0$. Engineers and physicists are frequently interested in integrals of functions (they represent total mass, momentum...), and without hesitation they would write

$$(1.3) \quad \lim_{n \to \infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n \to \infty} f_n(x)dx.$$  

In this case,

$$\lim_{n \to \infty} \int_0^1 f_n(x)dx = \lim_{n \to \infty} \left[ -\frac{1}{2} e^{-nx^2} \right]_0^1 = \frac{1}{2},$$
while
\[
\int_0^1 \lim_{n \to \infty} f_n(x) \, dx = \int_0^1 \lim_{n \to \infty} 0 \, dx = 0.
\]
That is, \(1/2 = 0\) - the order of integration and limit matters.

**Example 1.6.** In case you might think (1.3) is never true, consider \(f_n(x) = \frac{x^2}{n} + x\) on \([0, 1]\). Here \(\lim_{x \to \infty} f_n(x) = x\), and one checks that
\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \frac{1}{2} = \int_0^1 \lim_{n \to \infty} f_n(x) \, dx
\]
(check this!).

In early calculus courses students spend much time sketching graphs of functions. Even this is not so simple.

**Example 1.7.** Sketch the graph of
\[
f(x) = \begin{cases} 
1 & x \in \mathbb{Q}, \\
0 & x \in \mathbb{Q}^c
\end{cases}
\]
As we will see in Example 4.7, this function is not continuous anywhere. In this case, what does the area under the curve mean?

**Example 1.8.** (Cantor’s ternary set) You might think of subsets of \(\mathbb{R}\) as sets like \([0, 1]\) or \([-\pi, 0) \cup (2, 3)\). Subsets of \(\mathbb{R}\) can be complicated. Let \(F_1 = [0, 1]\), the unit interval. Now remove the open middle third and set \(F_2 = [0, 1/3] \cup [2/3, 1]\). If we remove the middle third in each of these intervals, we obtain \(F_3 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]\). Continue doing this indefinitely by removing the middle thirds of the remaining intervals each iteration. The Cantor Set is the intersection of the sets \(F_n\). That is,
\[
F = \bigcap_{n=1}^{\infty} F_n.
\]

It might appear at first that we removed all of the points in \([0, 1]\). Indeed, its length is in some sense zero since the length of \(F_1\) is \(|F_1| = 1\), \(|F_2| = 2/3\), \(|F_3| = 4/9\), and in general the length of \(F_n\) is \(|F_n| = (2/3)^n\). However \(F\) is not empty since 0, 1/3, 2/3, 1 \(\in F\). In fact \(F\) as an uncountable number of points and is equinumerous with \(\mathbb{R}\). Imagine writing the numbers in \([0, 1]\) in decimal form base 3. The Cantor set is obtained by removing all of the numbers in which a 1 appears anywhere in the decimal expansion - it is indeed a strange set. When we talk about functions defined on a set, we should not just think of the domain of the function as intervals.

For the rest of these notes we carefully reconstruct calculus, starting with some reasonable assumptions about the real numbers. At the end of the journey return to these examples and see if the context in which you view them has completely changed.