Test #3
Solutions

1. (30 pts) Solve the following homogeneous heat equation with insulating boundary conditions, AND compute \( \lim_{t \to \infty} u(x, t) \). Evaluate all constants.

\[
\frac{\partial u}{\partial t} = u_{xx} - u, \quad 0 < x < 1, \quad t > 0,
\]

\[
u_x(0, t) = 0, \quad u_x(1, t) = 0, \quad t > 0,
\]

\[
u(x, 0) = x, \quad 0 < x < 1.
\]

**Solution:** Separating the variables gives

\[
\frac{T'}{T} + 1 = \frac{X''}{X} = -\lambda^2.
\]

The solution to the \( T \) equation is \( T(t) = e^{-(\lambda^2 + 1)t} \). Since \( X'(0) = 0 \) and \( X'(1) = 0 \), the solution to the \( X \) equation is \( X_n(x) = \cos(n\pi x) \) with \( \lambda_n = n\pi, \quad n = 0, 1, 2, \ldots \). The general solution is

\[
u(x, t) = \sum_{n=0}^{\infty} a_n e^{-(\lambda^2 n^2 + 1)t} \cos(n\pi x)
\]

\[
= a_0 e^{-t} + \sum_{n=1}^{\infty} a_n e^{-(\lambda^2 n^2 + 1)t} \cos(n\pi x).
\]

The usual orthogonality trick shows

\[
a_0 = \int_0^1 x \, dx = \frac{1}{2}, \quad a_n = 2 \int_0^1 x \cos(n\pi x) \, dx = \frac{2}{n\pi} \left( \frac{1}{n^2\pi^2} - 1 \right).
\]

Finally, \( \lim_{t \to \infty} u(x, t) = 0 = \bar{u} \), and \( \bar{u} = 0 \) can easily be seen to solve the steady-state equation.

2. (30 pts)

(a) Find the steady-state solution, \( \bar{u} \), to

\[
\frac{\partial \bar{u}}{\partial t} = \bar{u}_{xx} + 2, \quad 0 < x < 1, \quad t > 0
\]

\[
u_x(0, t) = (u(0, t) - 1), \quad u(1, t) = 0, \quad t > 0,
\]

\[
u(x, 0) = x, \quad 0 < x < 1.
\]

(b) Set \( w(x, t) = u(x, t) - \bar{u}(x) \). Derive, but **Do Not** solve a PDE, boundary conditions, and initial conditions \( w \) satisfies.

**Solution:** The steady-state solution solves \( \bar{u}_{xx} = -2, \quad \bar{u}_x(0) = \bar{u}(0) - 1, \) and \( \bar{u}(1) = 0 \). That is, \( \bar{u} = -x^2 + ax + b \). Applying the boundary conditions we find \( a = 0 \) and \( b = 1 \). In particular, \( \bar{u} = -x^2 + 1 \).
Set \( w(x,t) = u(x,t) - \bar{u}(x) \). Then \( w_t = u_t, \ w_{xx} = u_{xx} - \bar{u}_{xx} = u_{xx} + 2 = u_t = w_t \), and \( w_t = w_{xx} \). Also, the boundary conditions show \( w_x(0,t) = u_x(0,t) - \bar{u}_x(0) = (u(t) - 1) - (\bar{u}(0) - 1) = u(0,t) - \bar{u}(0) = w(0,t) \). Similarly, \( w(1,t) = 0 \). The initial data becomes \( w(x,0) = u(x,0) - \bar{u}(x) = x + x^2 - 1 \), and in summary, \( w \) solves

\[
\frac{\partial w}{\partial t} = w_{xx}, \quad 0 < x < 1, \ t > 0
\]

\[
w_x(0,t) = w(0,t), \ w(1,t) = 0, \quad t > 0,
\]

\[
u(x,0) = x, \quad 0 < x < 1.
\]

3. (30 pts) Find the solution to the wave equation in the semi-infinite domain

\[
u_{tt} = c^2 u_{xx}, \quad 0 < x, t,
\]

\[
u(0,t) = \alpha(t), \quad t > 0,
\]

\[
u(x,0) = f(x), \quad 0 < x,
\]

\[
u_t(x,0) = 0, \quad 0 < x.
\]

**Solution:** Starting with d’Alembert’s solution, \( u(x,t) = A(x+ct) + B(x-ct) \), apply the initial data to find \( f(x) = A(x) + B(x) \). The initial data on the velocity implies \( 0 = cA'(x) - cB'(x) \). Integrating we find \( D = A(x) - B(x) \). Solving the two simultaneously, \( A(x) = \frac{D}{2} f(x) + \frac{D}{2} \) and \( B(x) = \frac{D}{2} f(x) + \frac{D}{2} \). Thus

\[
u(x,t) = \left( \frac{D}{2} f(x+ct) + \frac{D}{2} \right) + \left( \frac{D}{2} f(x-ct) - \frac{D}{2} \right)
\]

\[
eq \frac{f(x+ct) + f(x-ct)}{2} \quad x > ct.
\]

To find a formula for \( x < ct \) we apply the boundary condition. Indeed, \( u(0,t) = \alpha(t) = A(ct) + B(-ct) \). That is, \( B(-x) = \alpha \left( \frac{x}{c} \right) - A(x) \). Therefore, when \( x < ct \), \( B(x-ct) = \alpha \left( \frac{ct-x}{c} \right) - A(ct-x) \), and

\[
u(x,t) = \begin{cases} 
\frac{f(x+ct) + f(x-ct)}{2} & x > ct \\
\frac{f(x+ct) - f(x-ct)}{2} + \alpha \left( \frac{ct-x}{c} \right) & x < ct.
\end{cases}
\]

4. (b) and (d) below cannot be solved using the separation of variables method because they are nonlinear.

a) \( (t+1) \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \)

b) \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x} \)

c) \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \)

d) \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u^3 \).