Fun Problem Using Divergence Theorem

1. (Application of divergence theorem) Let $\Omega$ be a bounded, simply connected set in $\mathbb{R}^3$. Find a relation between $f$ and $g$ that is necessary for the PDE

$$\nabla^2 u = f, \quad x \in \Omega,$$

$$\frac{\partial u}{\partial n} = \nabla u \cdot n = g, \quad x \in \partial \Omega,$$

to have a solution. Try to interpret the result physically. Hint: integrate both sides of the PDE, $\nabla^2 u = f$, and use the boundary conditions.

2. Derive the two-dimensional divergence theorem

$$\int_S \int \nabla \cdot \mathbf{F} \, dA = \oint_{\partial S} \mathbf{F} \cdot n \, ds$$

from Green’s theorem. (This is not hard. Just write down the two-dimensional divergence theorem and Green’s theorem and compare the two.)

3. (Integration by parts) Suppose $\Omega$ is a simply connected domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. If the scalar $v$ is zero on $\partial \Omega$ show that

$$\int \int_\Omega v \nabla^2 u \, dV = - \int \int_\Omega \nabla v \cdot \nabla u \, dV.$$  

Hint: notice $\nabla \cdot (v \nabla u) = v \nabla^2 u + \nabla v \cdot \nabla u$.

4. From the integral form of Faraday’s law,

$$\oint_{\partial S} \mathbf{E} \cdot d\mathbf{R} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{A},$$

derive the Maxwell’s equation $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. 