Relations

In this chapter we generalize the notion of equality, of inequality, and carefully define the meaning of a function from a set $A$ to a set $B$. An examination of these three notions shows they involve two, typically, numbers ($a = b$, $a \leq b$, $f(a) = b$). We wish to extend the concepts to any set (not just real numbers). This, in part, motivates our first definition.

**Definition 4.1.** Suppose $A$ and $B$ are sets. The *Cartesian Product* of $A$ and $B$, denoted $A \times B$ is the set all ordered pairs $(a, b)$ in which the first coordinate is an element of $A$ and the second is an element of $B$. That is,

$$A \times B := \{(a, b) \mid a \in A, b \in B\}.$$

**Example 4.2.** Consider the sets $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Then

$$A \times B = \{(1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5)\}.$$

We will frequently need to show two sets are equal (have the same elements). Here is one way to accomplish this.

**Definition 4.3.** We say two sets $A$ and $B$ are equal, and write $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

**Theorem 4.4.** Suppose $A$ and $B$ are two non empty sets. Then $A \times B = B \times A$ if and only if $A = B$.

**Proof.** Suppose $A \times B = B \times A$. For any $x \in A$ and $y \in B$, we have $(x, y) \in A \times B$. Thus $(x, y) \in B \times A$. That is, $x \in B$ and $y \in A$. This implies $A = B$.

Conversely, suppose $A = B$. Then

$$A \times B = A \times A = B \times A$$

and the result follows.
Now we are in a position to define a relation which leads the generalizations we seek.

**Definition 4.5.** Suppose $A$ and $B$ are sets. Then $R \subseteq A \times B$ is called a relation from $A$ to $B$. That is, any subset, $R$, of $A \times B$ is a Relation from $A$ to $B$. If $B = A$, we just say $R$ is a relation on $A$ when $R \subseteq A \times A$.

**Example 4.6.** Again consider the sets $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. If we set
$$ R = \{(1,4),(2,4)\}, $$
then $R$ is a relation from $A$ to $B$.

We need a bit more structure before proceeding.

**Definition 4.7.** If $R$ is a relation from $A$ to $B$. We define the inverse relation to be
$$ R^{-1} := \{(b,a) \mid (a,b) \in R\}. $$

Notice that $R^{-1}$ is a relation from $B$ to $A$ since it is a subset of $B \times A$. We can compose relations too.

**Definition 4.8.** If $R$ is a relation from $A$ to $B$ and $S$ is a relation from $B$ to $C$, then the composite relation is
$$ S \circ R := \{(a,c) \in A \times C \mid (\exists b \in B)((a,b) \in R \land (b,c) \in S)\}. $$

Note $S \circ R$ is a relation from $A$ to $C$ (a subset of $A \times C$).

**Theorem 4.9.** Suppose $A$, $B$, and $C$ are non empty sets with $R$ a relation from $A$ to $B$ and $S$ a relation from $B$ to $C$. Then
\begin{enumerate}
  \item $(R^{-1})^{-1} = R$
  \item $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$
\end{enumerate}

**Proof.** We apply Definition 4.3 to the sets involved. To prove $a)$ suppose $(a,b) \in R$. Then by definition $(b,a) \in R^{-1}$. The definition of $(R^{-1})^{-1}$ is $(R^{-1})^{-1} = \{(b,a) \mid (b,a) \in R^{-1}\}$. Thus we see $(a,b) \in (R^{-1})^{-1}$ and so $R \subseteq (R^{-1})^{-1}$. Similar arguments show any $(a,b) \in (R^{-1})^{-1}$ is also in $R$. That is, $(R^{-1})^{-1} \subseteq R$, and part $a)$ follows.

To prove $b)$, suppose $(c,a) \in (S \circ R)^{-1}$. Then by definition of the inverse relation, $(a,c) \in (S \circ R)$. By the definition of the composite relation, there exits $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. This of course means $(c,b) \in S^{-1}$ and $(b,a) \in R^{-1}$. Applying again the definition of the composite relation, we find $(c,a) \in R^{-1} \circ S^{-1}$. Thus $(S \circ R) \subseteq R^{-1} \circ S^{-1}$. Similar arguments show $R^{-1} \circ S^{-1} \subseteq (S \circ R)$ and the result follows. \hfill $\square$

**Example 4.10.** Suppose $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, $C = \{7, 8, 9\}$, and
$$ R = \{(1,4),(1,5),(2,5)\}, \quad S = \{(4,7),(5,7),(6,9)\}. $$
Then $S \circ R = \{(1,7),(2,7)\}$, $(S \circ R)^{-1} = \{(7,1),(7,2)\}$. Moreover, $S^{-1} = \{(7,4),(7,5),(9,6)\}$ and $R^{-1} = \{(4,1),(5,1),(5,2)\}$. Thus $R^{-1} \circ S^{-1} = \{(7,1),(7,2)\}$ in accordance with the previous theorem.
**Definition 4.11.** Let $A$ be a nonempty set. We define the identity relation $i_A$ to be

$$i_A := \{(x, y) \in A \times A \mid x = y\}.$$ 

Thus, for example, if $A = \{1, 2, 3\}$, then $i_A = \{(1, 1), (2, 2), (3, 3)\}$.

**Definition 4.12.** Let $R$ be a relation on a nonempty set $A$.

1. $R$ is reflexive if and only if, for all $x \in A$, $(x, x) \in R$.
2. $R$ is symmetric if and only if, for all $x, y \in A$, if $(x, y) \in R$, then $(y, x) \in R$.
3. $R$ is transitive if and only if, for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

**Example 4.13.** Suppose $A = \{1, 2, 3\}$. Then

1. $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$ is reflexive but not symmetric or transitive.
2. $\{(1, 2), (2, 1)\}$ is symmetric but not reflexive or transitive.
3. $\{(1, 2), (2, 3), (1, 3)\}$ is transitive but not reflexive or symmetric.

**Example 4.14.** Suppose $A$ is a nonempty set and $R$ is a reflexive relation on $A$. Then $R \subseteq R \cap R$.

**Proof.** Suppose $(x, y) \in R$. Since $R$ is reflexive, $(y, y) \in R$. This implies, by the definition of $R \cap R$, $(x, y) \in R \cap R$. Thus $R \subseteq R \cap R$. $\square$

**Example 4.15.** Suppose $A$ is a nonempty set and $R_1, R_2$ are two transitive relations on $A$. Then $R_1 \cap R_2$ is transitive.

**Scratch work** The theorem is in the form $P \to Q$. However, here $Q$, the statement $R_1 \cap R_2$ is transitive, has the form

$$(\forall x, y, z \in A)[(x, y) \in R_1 \cap R_2 \land (y, z) \in R_1 \cap R_2 \to (x, z) \in R_1 \cap R_2].$$

It not only has a universal quantifier but has the form $P \to Q'$. Following the proof guidelines at the end of Chapter 3, the proof should start: Suppose $x, y, z \in A$ and $(x, y) \in R_1 \cap R_2$ and $(y, z) \in R_1 \cap R_2$. It concludes with $(x, z) \in R_1 \cap R_2$.

**Proof.** Let $x, y, z \in A$ and suppose $(x, y) \in R_1 \cap R_2$ and $(y, z) \in R_1 \cap R_2$. This means $(x, y) \in R_1$ and $(y, z) \in R_1$. Since $R_1$ is transitive, $(x, z) \in R_1$. A similar argument shows $(x, z) \in R_2$. Thus $(x, z) \in R_1 \cap R_2$, and $R_1 \cap R_2$ is transitive. $\square$

### 4.1. Equivalence Relations

In this section we generalize the notion of equality. If $a$ and $b$ are real numbers and $a = b$, we notice the equality sign is binary—it requires two numbers—like the Cartesian Product. We start by collecting some of the obvious properties of the equal sign. We expect $a = a$ for all real $a$. Moreover, if $a = b$, we certainly expect $b = a$. And finally, if $a = b$ and $b = c$, then $a = c$. This motivates the following definition.
Definition 4.16. Let $A$ be a nonempty set. Then $R \subseteq A \times A$ is an equivalence relation on $A$ if and only if

1. for all $a \in A$, $(a, a) \in R$,
2. if, for any $a, b \in A$, $(a, b) \in R$, then $(b, a) \in R$,
3. if, for any $a, b, c \in A$, $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

As in the previous section the first property is called reflexive; the second symmetric, and the third is called transitive.

To simplify the notation, and make the generalization look more like an equal sign, instead of writing $(a, b) \in R$, with $R$ an equivalence relation, we write $a \sim b$.

Thus we can restate the definition as

Definition 4.17. The binary relation $\sim$ on $A$ is said to be an equivalence relation on $A$ if, for all $a, b, c \in A$, we have

1. (Reflexive) $a \sim a$,
2. (Symmetric) $a \sim b$ implies $b \sim a$,
3. (Transitive) $a \sim b$ and $b \sim c$ implies $a \sim c$.

Example 4.18. Let $A = \{1, 2, 3, 4\}$. We claim the following are equivalence relations on $A$.

a) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

b) $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$

c) $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (2, 4), (4, 2), (4, 1), (1, 4)\}$.

Example 4.19. Suppose $A = \mathbb{Z}$ the set of all integers. For $a, b \in A$ we say $a \sim b$ ($a$ is equivalent to $b$) if $a$ and $b$ have the same parity (both even or both odd). Show this is an equivalence relation on $A$.

Solution. So show this, we apply the Definition 4.17. Note that $a$ and $b$ have the same parity iff $a - b$ is even.

1. Let $a \in A$. Then $a - a = 0$. Since 0 is even, $a \sim a$ for any $a \in A$ and the relation is reflexive.
2. Suppose $a \sim b$ for two elements in $A$. Then $a - b$ is even. It follows $-(a - b) = b - a$ is even. That is, $b \sim a$ and the relation is symmetric.
3. Suppose for some $a, b, c \in A$, $a \sim b$ and $b \sim c$. Then both $a - b$ and $b - c$ are even. Hence $a - c = (a - b) + (b - c)$ is even. That is, $a \sim c$ and the relation is transitive.

This shows that $\sim$ is an equivalence relation on $\mathbb{Z}$.

With a few more definitions more fun can be had.

Definition 4.20. Suppose $A$ is a set and $\sim$ is an equivalence relation on $A$. The equivalence classes of $a \in A$, denoted $[a]$, is the set

$[a] := \{x \in A \mid a \sim x\}$.

The set of all equivalent classes is denoted $A/R$. 


Example 4.21. Find all the equivalent classes in Example 4.19.

Solution. Let us compute \([0]\). It is the set \(\{x \in \mathbb{Z} \mid x \sim 0\}\). This is the same as \(x - 0 = 2k\) for some \(k \in \mathbb{Z}\). That is, \([0]\) is all \(x\) such that \(x\) is even. Thus \([0] = \{2k \mid k \in \mathbb{Z}\}\). A similar calculation shows \([2] = [0]\). In fact, the equivalence class of any even number is the even numbers.

Now we compute \([1]\). This would be all \(x \in \mathbb{Z}\) such that \(1 - x = 2k\) for some \(k \in \mathbb{Z}\). That is, \(x\) has the form \(x = 2k + 1\) for some \(k \in \mathbb{Z}\), and \([1]\) is all of the odd numbers. In particular \([1] = [3] = [5] \ldots\).

Finally, note that \(\mathbb{Z} = [0] \cup [1]\) and \([0] \cap [1] = \emptyset\). That is, \([0]\) and \([1]\) have no common elements and provide a partition of the original set. Here \(A/R = \{[0], [1]\}\).

Example 4.22. Let \(A = \mathbb{R}\). We say \(\theta_1 \sim \theta_2\) if \(\theta_1 - \theta_2 = 2k\pi\) for some \(k \in \mathbb{Z}\).

Check this is an equivalence relation on \(A\) and find the equivalent classes.

Solution.

(1) We check the relation is reflexive: Let \(\theta \in A\). Then \(\theta - \theta = 0 = 2 \cdot 0\pi\) and thus \(\theta \sim \theta\) for any \(\theta \in A\).

(2) Next we show the relation is symmetric. Suppose \(\theta_1 \sim \theta_2\) for two elements in \(A\). Then \(\theta_1 - \theta_2 = 2k\pi\) for some \(k \in \mathbb{Z}\). It follows \(\theta_2 - \theta_1\) has the same form and relation is symmetric.

(3) Finally we show the relation is transitive. Suppose for some \(\theta_1, \theta_2, \theta_3 \in A\), \(\theta_1 \sim \theta_2\) and \(\theta_2 \sim \theta_3\). Then \(\theta_1 - \theta_3 = (\theta_1 - \theta_2) + (\theta_2 - \theta_3)\). Under our assumptions, we find \(\theta_1 - \theta_3 = 2(k_1 + k_2)\pi\) for some \(k_1, k_2 \in \mathbb{Z}\), and the relation is transitive.

To find the equivalence classes we simply compute a few to see the pattern. Note

\([0] = \{0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots\}\)  
\([\pi/4] = \{\pi/4, \pi/4 \pm 2\pi, \pi/4 \pm 4\pi, \pi/4 \pm 6\pi, \ldots\}\)  
\([\pi/2] = \{\pi/2, \pi/4 \pm 2\pi, \pi/2 \pm 4\pi, \pi/2 \pm 6\pi, \ldots\}\)

\(\vdots\)

Again we see \([\theta_1] \cap [\theta_2] = \emptyset\) when \(\theta_1 \not\sim \theta_2\) and \([\theta_1] = [\theta_2]\) when \(\theta_1 \sim \theta_2\). The union of all the equivalence classes is all of \(\mathbb{R}\) - the equivalences classes form a partition of \(\mathbb{R}\).

Example 4.23. Let \(A\) be the points in the plane \((A = \mathbb{R} \times \mathbb{R})\). We say two points are equivalent if they are equal distance from the origin. So \((x, y) \sim (w, z)\) if \(x^2 + y^2 = w^2 + z^2\). Show this is an equivalence relation on \(A\) and find the equivalent classes.

Solution.
(1) (Reflexive) Let \((x, y) \in A\). Then, \((x, y) \sim (x, y)\) since \(x^2 + y^2 = x^2 + y^2\).

(2) (Symmetric) Suppose two elements in \(A\) are related: \((x, y) \sim (w, z)\). Then \(x^2 + y^2 = w^2 + z^2\). Obviously this implies \(w^2 + z^2 = x^2 + y^2\) and so \((w, z) \sim (x, y)\).

(3) (Transitive) Suppose for some \((x_1, y_1) \in A\), \((x_2, y_2) \in A\) and \((x_3, y_3) \in A\) we have \((x_1, y_1) \sim (x_2, y_2)\) and \((x_2, y_2) \sim (x_3, y_3)\). By assumption \(x_1^2 + y_1^2 = x_2^2 + y_2^2\) and \(x_2^2 + y_2^2 = x_3^2 + y_3^2\). Thus \((x_1, y_1) \sim (x_3, y_3)\).

The equivalence class of, say, \([1, 0]\) would be all points such that \(x^2 + y^2 = 1^2 + 0^1 = 1\). That is, all points on the circle centered at the origin passing through the point \((1, 0)\). Thus we see the equivalence classes are concentric circles centered at the origin. The circles are disjoint and their union is the entire plane.

We tie these ideas together with the following theorem.

**Theorem 4.24.** Let \(A\) be a non empty set. If \(R\) is an equivalence relation on \(A\), the distinct equivalence classes provide a decomposition of \(A\) into mutually disjoint subsets. Conversely, given any decomposition of \(A\) into disjoint subsets, there exists an equivalence relation on \(A\) for which these subsets are the distinct equivalent classes.

Before proceeding to the proof, we better define what we mean by union of sets. We define the intersection of collection of sets as well.

**Definition 4.25.** Let \(I\) be any non empty set and let \(A_i\) be a set for each \(i \in I\). The set \(I\) is called an index set. We define
\[
\bigcup_{i \in I} A_i = \{x | (\exists i \in I)(x \in A_i)\},
\]
and
\[
\bigcap_{i \in I} A_i = \{x | (\forall i \in I)(x \in A_i)\}.
\]

The theorem also claims the equivalence classes are disjoint. We better check that.

**Lemma 4.26.** Suppose \(A\) is a nonempty set and \(R\) is an equivalence relation on \(A\). Then \(x \sim y\) if and only if \([x] = [y]\). Moreover, if \(x \not\sim y\), then \([x] \cap [y] = \emptyset\).

**Proof.** Suppose \(x \sim y\) and \(z \in [x]\). Then, by definition of an equivalence class, \(z \sim x\). By transitivity, \(z \sim y\). This means \(z \in [y]\) and \([x] \subseteq [y]\). A similar argument shows \([y] \subseteq [x]\).

Conversely, suppose \([x] = [y]\). Note \(x \in [x]\) since \(x \sim x\) (reflexivity). Hence \(x \in [y]\). That is, \(x \sim y\).

We prove the contrapositive of the last statement. Suppose \([x] \cap [y] \neq \emptyset\). Then \(z \in A\) exists so that \(z \in [x]\) and \(z \in [y]\). As we have seen in the above arguments, this implies \(x \sim y\).
4.2. Partial and Total Orders

**Proof.** (Proof of Theorem 4.24.) Let $R$ be an equivalence relation on $A$. Since $a \in [a]$ for all $a \in A$, we have $\bigcup_{a \in A} [a] = A$. The previous lemma shows the elements of $A/R$ are disjoint.

To see the converse, let $A$ be any set with $\bigcup_{i \in I} A_i = A$ for some index set $I$ and set $A_i$ with $A_i \cap A_j = \emptyset$ for $i \neq j$. We say $a \sim b$ if $a, b \in A_i$ for some $i \in I$. We can easily check this an equivalence relation on $A$. Moreover, by construction the equivalence classes are the $A_i$. \qed

### 4.2. Partial and Total Orders

In this section we generalize the notion of inequality.

**Definition 4.27.** Let $A$ be a nonempty set. Then $R \subseteq A \times A$ is called a **partial order** if it is reflexive, antisymmetric, and transitive. That is,

1. for all $a \in A$, $(a, a) \in R$.
2. for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.
3. for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

The relation is called a **total order** if, in addition,

4. for all $a, b \in A$ either $(a, b) \in R$ or $(b, a) \in R$.

To simplify the notation we may sometimes write $a \preceq b$ when $(a, b) \in R$.

**Example 4.28.** If $A = \mathbb{R}$ and $(a, b) \in R$ means $a \leq b$, then $R$ is a total order.

**Example 4.29.** Let $A$ be the set of all English words. We say $a \preceq b$ if $a$ appears before $b$ in the dictionary. One can check this is generates a total order on $A$.

**Example 4.30.** Let $S$ be a set with at least two elements. Set $A := \mathcal{P}(S) = \{B \mid B \subseteq S\}$, the set of all subsets of $S$. We say, for $a, b \in \mathcal{P}(S)$, $a \preceq b$ if $a \subseteq b$. One can check this is a partial order. It is not a total order. Indeed, if $a_1, a_2 \in A$ with $\{a_1\} \neq \{a_2\}$, then 4. above is not satisfied.

**Example 4.31.** Let $A = \mathbb{N}$ We say $a \preceq b$ if $b/a \in \mathbb{N}$. This is produces a partial order which is not a total order on $A$. Indeed,

1. for all $n \in \mathbb{N}$, $n/n = 1 \in \mathbb{N}$. Thus $n \preceq n$ and $R$ is reflexive.
2. Suppose $a, b \in \mathbb{N}$. If $a/b \in \mathbb{N}$ and $b/a \in \mathbb{N}$, then $a/b = n_1$ and $b/a = n_2$ for some $n_1, n_2 \in \mathbb{N}$. This implies $1/n_1 = n_2$ or $n_1n_2 = 1$ This in turn implies $a = b$ and $R$ is antisymmetric.
3. Suppose $a, b, c \in \mathbb{N}$, $b/a \in \mathbb{N}$, and $c/b \in \mathbb{N}$. Then $(b/a)(c/b) = n_1n_2$ for some $n_1, n_2 \in \mathbb{N}$. That is, $c/a \in \mathbb{N}$ and $R$ is transitive.

It is not total order since, for example, $(7, 2) \notin R$.

Now that we have a notion of ordering we can define a minimum and maximum. However, the next example shows this is not so straightforward.
Example 4.32. Let \( S = \{1, 2, 3\} \). Set 
\[ A = \mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}. \]
As in an above example, we say, for \( a, b \in \mathcal{P}(S) \), \( a \preceq b \) if \( a \subseteq b \). Now set 
\[ B = \{\{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}. \]
So \( B \subseteq A \). We ask - does \( B \) have a minimum or smallest element? We see both \( \{1\} \) and \( \{2\} \) are smaller than \( \{1, 2\} \) and \( \{1, 2, 3\} \), but \( \{1\} \) and \( \{2\} \) are not ordered-so there is no minimum element. However, we notice there are no elements in \( B \) smaller than \( \{1\} \) and \( \{2\} \). This suggests the following definitions.

We could define a minimum as we did for real numbers:

**Definition 4.33.** Suppose \( A \) is a nonempty set, \( R \) is a partial order on \( A \), and \( B \subseteq A \). Let \( b \in B \). Then \( b \) is the minimum of \( B \) if

(i) \( b \in B \),
(ii) \( b \preceq x \) for all \( x \in B \).

We see from the previous example no such \( b \) exists. So we might try to loosen the restrictions on \( b \) a bit.

**Definition 4.34.** Suppose \( A \) is a nonempty set, \( R \) is a partial order on \( A \), and \( B \subseteq A \). Let \( b \in B \). Then \( b \) is the smallest element of \( B \) if

(i) \( b \in B \),
(ii) it is false there exists \( x \in B \) such that \( x \preceq b \) and \( x \neq b \).

In the first definition, the definition of a minimum, \((ii)\) requires that \( b \) can be compared to all \( x \in B \). In a partial order there may be no relation between some elements. The second definition, of a smallest element, does not require \( b \) to be compared to all other elements in \( B \) - it just requires that there are no smaller elements. Thus we might expect the two definitions to be the same if the partial order is a total order. Also, note we cannot negate \( x \preceq b \) in the Definition 4.34 in the case \( R \) is a partial order since we do not know enough about the ordering. However, if \( R \) is a total order we can (see homework problem (4.21)).

**Example 4.35.** Returning to Example 4.32, we see both \( \{1\} \) and \( \{2\} \) are smallest elements.

**Example 4.36.** We might hope that at every non empty set has either a smallest or minimum element. This, however, is not the case. Consider \( A = \mathbb{R} \) with \( a \preceq B \) defined to be the usual inequality \( a \leq b \). Let \( B := \{x \mid x > 2\} \). Then \( B \) has neither a minimum or a smallest element.
4.3. Functions

Like equivalence relations and partial orders, a function is also simply a relation.

**Definition 4.37.** Suppose $F$ is a relation from $A$ to $B$. Then $F$ is called a function from $A$ to $B$ if, for all $a \in A$, there is a unique $b \in B$ such that $(a, b) \in F$.

When the relation $F$ is a function we write $F : A \to B$. Moreover, if $(a, b) \in F$, we write $f(a) = b$.

We note that we are apparently assuming the domain of the function is all of $A$. We also emphasize the notation $F : A \to B$ is a fancy way of writing a relation from $A$ to $B$ - that is, it just represents a subset of $A \times B$ with a certain property.

**Theorem 4.38.** Suppose $F : A \to B$ and $G : B \to C$. Then $G \circ F : A \to C$ and for any $a \in A$, $(g \circ f)(a) = g(f(a))$.

**Proof.** By definition $G \circ F$ is a relation on $A \times C$. We need to show the relation is a function. That is, for all $a \in A$ we need to show there is an unique $c \in C$ such that $(a, c) \in G \circ F$.

Existence: Let $a \in A$. Since $F$ is a function, $b \in B$ exists such that $(a, b) \in F$. Since $G$ is a function, $c \in C$ exists such that $(b, c) \in G$. By definition of $G \circ F$, $(a, c) \in G \circ F$. That is, for each $a \in A$, $c \in C$ exists such that $(a, c) \in G \circ F$.

Uniqueness: We need to show there is only one such $c$. Suppose $c_1, c_2 \in C$ exists so that $(a, c_1), (a, c_2) \in G \circ F$. By definition of the composite relation, $b_1, b_2 \in B$ exists so that $(a, b_1) \in F$, $(b_1, c_1) \in G$ and $(a, b_2) \in F$, $(b_2, c_2) \in G$. Since $F$ is a function $b_1 = b_2$. Since $G$ is function $c_1 = c_2$. Thus $G \circ F : A \to C$ ($G \circ F$ is a function).

To show the last part, recall we write $b = f(a)$ when $(a, b) \in F$. Thus we have $(g \circ f)(a) = c$. Since $(b, c) \in G$, $g(b) = c$ and similarly $f(a) = b$. That is, $(g \circ f)(a) = g(f(a))$. \hfill $\Box$

**Definition 4.39.** Suppose $F : A \to B$.

(a) We say $F$ is injective or one-to-one, iff for all $a_1, a_2 \in A$, $b \in B$, $(a_1, b) \in F$ and $(a_2, b) \in F$ implies $a_1 = a_2$ (i.e. $f(a_1) = f(a_2)$ implies $a_1 = a_2$).

(b) We say $F$ is surjective or onto iff for all $b \in B$ there exists $a \in A$ such that $(a, b) \in F$ (or there exists $b \in B$ such that $f(a) = b$).

(c) If a function is injective and surjective, it is said to be bijective.

Using our negating skills we should be able to prove

**Lemma 4.40.** Let $F : A \to B$.

(a) The function $F$ is not injective if there exist $a_1, a_2 \in A$ and $b \in B$ with $a_1 \neq a_2$ such that $(a_1, b) \in F$ and $(a_2, b) \in F$ (or $f(a_1) = f(a_2)$).

(b) The function $F$ is not surjective if there exists $b \in B$ such that for all $a \in A$, $(a, b) \notin F$. 
Proof. In quantifier notation injectivity is $((∀a_1, a_2 ∈ A)((a_1, b) ∧ (a_2, b) ∈ F → a_1 = a_2))$. Its negation is

$((∃a_1, a_2 ∈ A)(∃b ∈ B)\!(\!(a_1, b) ∧ (a_2, b) ∈ F) ∧ (a_1 ≠ a_2))$.

In quantifier notation surjectivity is $((∀b ∈ B)(∃a ∈ A)((a, b) ∈ F))$. Its negation is

$((∃b ∈ B)(∀a ∈ A)((a, b) \notin F))$.

**Theorem 4.41.** Suppose $F: A → B$ and $G: B → C$.

(a) If $F$ and $G$ are injective, so is $G \circ F$.
(b) If $F$ and $G$ are surjective, so is $G \circ F$.

Proof. We first note that Theorem 4.38 shows $G \circ F$ is a function from $A$ to $C$.

To prove (a) suppose $a_1, a_2 ∈ A$, $c ∈ C$, $(a_1, c) ∈ G \circ F$ and $(a_2, c) ∈ G \circ F$. We need to show $a_1 = a_2$. It follows $b_1, b_2 ∈ B$ exist so that $(a_1, b_1) ∈ F$, $(b_1, c) ∈ G$, and $(a_2, b_2) ∈ F$, $(b_2, c) ∈ G$. Since $G$ is a injective $b_1 = b_2$. We see now that $(a_1, b_1) ∈ F$ and $(a_2, b_1) ∈ F$. Since $F$ is injective, $a_1 = a_2$.

To prove (b), let $c ∈ C$ be any element in $C$. Since $G$ is surjective, there exists a $b ∈ B$ such that $(b, c) ∈ G$. Since $F$ is surjective, there exists $a ∈ A$ such that $(a, b) ∈ F$. By definition of composition $(a, c) ∈ G \circ F$. That is, $G \circ F$ is surjective.

**Theorem 4.42.** Suppose $F: A → B$. If $F$ is bijective, then $F^{-1}: B → A$. Moreover, $F^{-1}$ is a bijection.

Proof. The theorem asserts the inverse relation of $F$ is a function. So we must check this. We know $F^{-1}$ is a relation from $B$ to $A$. Let $b ∈ B$. Since $F$ is surjective, there is a $a ∈ A$ such that $(a, b) ∈ F$. Thus $(b, a) ∈ F^{-1}$ for each $b ∈ B$. We need to show the $a$ is unique. Thus, suppose $(b, a_1) ∈ F^{-1}$ and $(b, a_2) ∈ F^{-1}$ for some $a_1, a_2 ∈ A$. By the definition of the inverse relation, $(a_1, b)$ and $(a_2, b)$ are in $F$. Since $F$ is injective $a_1 = a_2$. This shows $F^{-1}$ is a function.

To show $F^{-1}$ is a bijection, we first show $F^{-1}$ is onto. Suppose $a ∈ A$. Since $F$ is a function, there exists $b ∈ B$ such that $(a, b) ∈ F$. That is, $b ∈ B$ exists so that $(b, a) ∈ F^{-1}$. This shows $F^{-1}$ is onto. Next we show $F^{-1}$ is injective. Suppose, for $b_1, b_2 ∈ B$, $a ∈ A$, $(b_1, a), (b_2, a) ∈ F^{-1}$. Then $(a, b_1) ∈ F$ and $(a, b_2) ∈ F$. Since $F$ is a function $b_1 = b_2$ and $F$ is injective.

**Example 4.43.** Consider $F: [0, 1] → [1, 2]$ with $f(x) = x^2 + 1$ and $G: [1, 2] → [0, 1]$ with $g(x) = \sqrt{x - 1}$. Note that $f \circ g(x) = x$ on $[1, 2]$, while $g \circ f(x) = x$ on $[0, 1]$. Also note $g = f^{-1}$. The next theorem confirm this result in our abstract setting.

**Theorem 4.44.** Suppose $F: A → B$, $G: B → A$, $G \circ F = i_A$, and $F \circ G = i_B$. Then $G = F^{-1}$.

Proof. Suppose $(b, a) ∈ G$. Since $a ∈ A$ and $F$ is a function, $b_1 ∈ B$ exists so that $(a, b_1) ∈ F$. This means $(b, b_1) ∈ F \circ G$. Since $F \circ G = i_B$, $b_1 = b$. That is, $(a, b) ∈ F$. Hence, $(b, a) ∈ F^{-1}$ and $G ⊆ F^{-1}$.

Conversely, suppose $(b, a) ∈ F^{-1}$. Then $(a, b) ∈ F$. Since $G$ is a function a unique $a_1 ∈ A$ exists so that $(b, a_1) ∈ G$. This implies $(a, a_1) ∈ F \circ G$. Since $F \circ G = i_A$, $a = a_1$. This shows, $(b, a) ∈ G$. That is, $F^{-1} ⊆ G$, and $G = F^{-1}$.
4.4. Homework

Exercise 4.1. Suppose $A$ and $B$ are non-empty sets.

(a) If $R_1, R_2$ are two symmetric relations on $A$. Does it follow $R_1 \cup R_2$ is a symmetric relation on $A$? Give a proof or a counterexample.

(b) Suppose $R_1$ is an equivalence relation on $A$ and $R_2$ is an equivalence relation on $B$ with $A \cap B = \emptyset$. Prove $R_1 \cup R_2$ is an equivalence relation on $A \cup B$.

Exercise 4.2. Prove $R$ is transitive on a set $A$ if and only if $R^{-1}$ is transitive.

Exercise 4.3. Suppose $R$ is a relation on a non-empty set $A$ which is not transitive. Prove there exists smallest relation $R_T$ such that $R_T$ is transitive and $R \subseteq R_T$. In particular, if $S$ is any other transitive relation on $A$ and $R \subseteq S$, then $R_T \subseteq S$.

Exercise 4.4. For the given set and relation below determine which define equivalence relations.

(a) $S$ is the set of all people alive today, $a \sim b$ if $a$ and $b$ have the same father.

(b) $S$ is the set of real numbers, $a \sim b$ if $a = \pm b$.

(c) $S$ is the set straight lines in the plane, $a \sim b$ if $a$ and $b$ are parallel.

Exercise 4.5. If $R_1$ and $R_2$ are reflexive relations on a set $A$, prove $R_1 \circ R_2$ is reflexive on $A$.

Exercise 4.6. Suppose $R_1$ and $R_2$ are symmetric relations on a set $A$. Prove that $R_1 \circ R_2$ is symmetric if and only if $R_1 \circ R_2 = R_2 \circ R_1$.

Exercise 4.7. Suppose $R_1$ and $R_2$ are transitive relations on a set $A$. Prove that, if $R_2 \circ R_1 \subseteq R_1 \circ R_2$, then $R_1 \circ R_2$ is transitive.

Exercise 4.8. Let $A = \mathbb{N}$. Let $R$ be the relation characterized by $n \sim m$ if $|n - m|$ is divisible by 3. Show $R$ is an equivalence relation, find the equivalence classes, and find $A/R$.

Exercise 4.9. Find all of the equivalence relations on the set $A = \{1, 2, 3\}$.

Exercise 4.10. Negate (2) in Definition 4.16 (a symmetric relation) and give an example of a relation $R$ on $\mathbb{N}$ that is not symmetric but is reflexive and transitive.

Exercise 4.11. Define a relation $R$ on $\mathbb{Z}$ by $x \sim y$ if $3$ divides $x + 2y$. Prove $R$ is an equivalence relation and find $A/R$.

Exercise 4.12. Let $A$ be a set, $x \in A$, and $R$ an equivalence relation on $A$. Write all of the true statements that have the form $p \in q$ or $p \subseteq q$, where $p \neq q$ and $p$ and $q$ are taken from the following list of five items:

\begin{align*}
A & \quad A \times A & \quad \mathcal{P}(A) & \quad A/R & \quad [x]
\end{align*}

Exercise 4.13. Let $R$ be the relation on $A$. Prove the following:

(1) $R$ is reflexive iff $i_A \subseteq R$.

(2) $R$ is symmetric iff $R = R^{-1}$.
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(3) $R$ is transitive iff $R \circ R \subseteq R$.

**Exercise 4.14.** Let $A$ be a nonempty set and $R$ an equivalence relation on $A$. Suppose $B \subseteq A$. Set $S = R \cap (B \times B)$.

(a) Prove $S$ is an equivalence relation on $B$.
(b) Prove for all $x \in B$, $[x]_S = [x]_R \cap B$, where $[x]_S$ and $[x]_R$ are the equivalence classes with respect to the relation $S$ and $R$.

**Exercise 4.15.** Find all of the partial order relations on the set $A = \{a, b, c\}$, where $a, b, c$ are distinct elements.

**Exercise 4.16.** Which of the following are partial orders, total order, or neither.

(a) $A$ is the set of all English words. We say $x \preceq y$ (i.e. $(x, y) \in R$), if $y$ occurs at least as late in alphabetical order as the word $x$.
(b) $A$ is the set of all English words. We say $x \preceq y$ if the first letter of the word $y$ occurs at least as late in alphabet as the first letter of the word $x$.
(c) $A$ is the set of all countries on Earth. We say $x \preceq y$, if the population of country $y$ is at least as large as the population of the country $x$.

**Exercise 4.17.** Suppose $R$ is a relation on $A$. Prove that $R$ is both symmetric and antisymmetric iff $R \subseteq i_A$.

**Exercise 4.18.** Suppose $R$ is a partial order on $A$. Prove that $R^{-1}$ is a partial order on $A$. If $R$ is a total order, will $R^{-1}$ be a total order?

**Exercise 4.19.** Suppose $R_1$ and $R_2$ are partial orders on $A$. Give a proof or a counter example to the following.

(a) Is $R_1 \cap R_2$ a partial order on $A$?
(b) Is $R_1 \cup R_2$ a partial order on $A$?

**Exercise 4.20.** Suppose $R$ is a partial order on $A$ and $B \subseteq A$. Prove that $R \cap (B \times B)$ is a partial order on $B$.

**Exercise 4.21.** Suppose $A$ is a non-empty set, $R$ is a partial order on $A$, and $B \subseteq A$.

(a) If $B$ as a minimum $b$, show it is unique.
(b) Show by example that the smallest element, if it exists, may not be unique.
(c) Show that, if $b$ is a minimum, it is a smallest element and the only one.
(d) If $R$ is a total partial order and $B$ has a smallest element $b$, then $b$ is a minimum (i.e. the definitions of minimum and smallest element agree).

**Exercise 4.22.** Let $f$ and $g$ be functions from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$f(x) = \frac{1}{x^2 + 2}, \quad g(x) = 2x - 1.$$ 

Compute $(f \circ g)(x)$ and $(g \circ f)(x)$.

**Exercise 4.23.** Suppose $A$ is a non-empty set. Show that $i_A$ is the only relation on $A$ that is both an equivalence relation on $A$ and a function from $A$ to $A$. 

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(Continued on the next page)
Exercise 4.24. Let $F : A \to B$ and $S$ be a relation on $B$. Define a relation $R$ on $A$ as follows:

$$R = \{(x, y) \in A \times A \mid (f(x), f(y)) \in S\}.$$

Prove the following.
(a) If $S$ is reflexive, then so is $R$.
(b) If $S$ is symmetric, then so is $R$.
(c) If $S$ is transitive, then so is $R$.

Exercise 4.25. Suppose $A$ is a nonempty set and $F : A \to A$.
(a) Suppose there is some $a \in A$ such that, for all $x \in A$, $f(x) = a$. That is, $f$ is a constant function. Prove that for all $G : A \to A$, $F \circ G = F$.
(b) Suppose that for all $G : A \to A$, $F \circ G = F$. Prove that $F$ is a constant function.

(a) Suppose $G : A \to A$ and $R = \{(x, y) \in A \times A \mid g(x) = g(y)\}$. Show that $R$ is an equivalence relation on $A$.
(b) Suppose $R$ is an equivalence relation on $A$ and let $G : A \to A/R$ be defined by $g(x) = [x]$. Show that $R = \{(x, y) \in A \times A \mid g(x) = g(y)\}$.

Exercise 4.27. Let $A = \mathcal{P}(\mathbb{R})$. Define $F : \mathbb{R} \to A$ by $f(x) = \{y \in \mathbb{R} \mid y^2 < x\}$.
(a) Find $f(2)$.
(b) Is $f$ injective? Is it surjective?

Exercise 4.28. Suppose $F : A \to B$ and $G : B \to C$. Prove the following.
(a) If $G \circ F$ is onto, then $G$ is onto.
(b) If $G \circ F$ is injective, then $F$ is injective.

Exercise 4.29. Suppose $F : A \to B$ and $G : B \to C$. Prove the following.
(a) If $F$ is onto and $G$ is not injective, then $G \circ F$ is not injective.
(b) If $F$ is not surjective and $G$ is injective, then $G \circ F$ is not onto.

Exercise 4.30. Suppose $A$, $B$, and $C$ are sets, $F : A \to B$, $G : B \to C$, and $H : B \to C$. If $F$ is onto and $G \circ F = H \circ F$, then $G = H$.

Exercise 4.31. Suppose $R$ is an equivalence relation on $A$, and let $G : A \to A/R$ be defined by $g(a) = [a]$.
(a) Show $G$ is onto
(b) Show $G$ is injective iff $R = i_A$.

Note: The following are from MAT 300 tests given at ASU

Exercise 4.32. Let $F : A \to B$ be a function, and let $R$ be the relation on $A$ defined by

$$R = \{(a, c) \in A \times A \mid f(a) = f(c)\}.$$
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a) Prove $R$ is an equivalence relation on $A$.

b) Describe $[4]$ in the case $A = B = \mathbb{R}$ and $f(x) = x^2$.

c) Prove the following statement for general $F$. If $R = \{(a, a) \mid a \in A\}$, then $F$ is injective.

**Exercise 4.33.** Let $A = \{a, b, c, d\}$, and let $B = \{p, q\}$, and let $R = \{(a, p), (c, p), (c, q), (d, q)\} \subseteq A \times B$.

a) Is $R$ a function? Explain.

b) List the elements of $R^{-1}$.

c) List the elements of $R^{-1} \circ R$.

d) Is $R^{-1} \circ R$ symmetric? Explain.

**Exercise 4.34.** Let $F : \mathbb{R} \setminus \{5\} \to \mathbb{R} \setminus \{2\}$ be defined by $f(x) = \frac{2x + 5}{x - 5}$. Prove that $f$ is onto.

**Exercise 4.35.** Let $R$ be a relation from $A$ to $B$. Suppose that $R \circ R^{-1} \subseteq i_B$ and $i_A \subseteq R^{-1} \circ R$. Prove that $R$ is a function from $A$ to $B$.

**Exercise 4.36.** Define a relation $M$ of $\mathbb{Z}$ as follows: for integers $x$ and $y$, $x \sim y$ if there exists a natural number $k$ such that $x$ divides $y^k$, and $y$ divides $x^k$. Prove that $M$ is an equivalence relation.

**Exercise 4.37.** Let $F : A \to B$ be injective. Prove that for all $X \subseteq A$ and $Y \subseteq B$, if $Y \subseteq f(X)$, then $f^{-1}(Y) \subseteq X$.

**Exercise 4.38.** In each of the following, decide if the given relation is a function. Prove your answer.

a) $\{(x, y) \in [0, \infty) \times \mathbb{R} \mid x^2 = y^2 + 1\}$.

b) $\{(x, y) \in [0, \infty) \times \mathbb{R} \mid y^2 = x^2 + 1\}$.

**Exercise 4.39.** Let $F : A \to B$ and $G : B \to A$. True or false.

a) If $G$ is injective and $G \circ F$ surjective, then necessarily $G^{-1} : B \to A$.

b) If $F$ is injective and $G \circ F$ surjective, then necessarily $F^{-1} : B \to A$.

c) If $G \circ F$ is injective and $F \circ G$ surjective, then necessarily $F^{-1} : B \to A$.

d) If $F \circ G$ is injective and $G \circ F$ surjective, then necessarily $F^{-1} : B \to A$. 