systems

We consider systems of differential equations of the form

\[ x'_1 = ax_1 + bx_2, \]
\[ x'_2 = cx_1 + dx_2, \]

(1)

where \(a, b, c,\) and \(d\) are real numbers. At first glance the system may seem to be first-order; however, it is coupled, and this two-dimensional system is more closely related to a second-order differential equation.

Before finding the solution, some notation is introduced to simplify the form of the system. The new notation implicitly removes most of the = and + operations. The form of the system suggests arranging the unknown functions \(x_1, x_2\) vertically and enclosed in brackets (i.e. matrix form)

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\]

(2)

This new object is similar to a number or function. Indeed, operations may be performed on (2). Let \(a \in \mathbb{R}\). We define

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}' := \begin{pmatrix}
  u' \\
  v'
\end{pmatrix},
\]

\[
a \begin{pmatrix}
  u \\
  v
\end{pmatrix} := \begin{pmatrix}
  u \\
  v
\end{pmatrix} a := \begin{pmatrix}
  au \\
  av
\end{pmatrix}.
\]

The new object can be part of mathematical statements. We say

\[
\begin{pmatrix}
  a \\
  b
\end{pmatrix} = \begin{pmatrix}
  c \\
  d
\end{pmatrix}
\]

if and only if \(a = c\) and \(b = d\). Moreover, we set

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} := \begin{pmatrix}
  ax_1 + bx_2 \\
  cx_1 + dx_2
\end{pmatrix}.
\]

Then (1) is equivalent to

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}' = \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix},
\]

often written more simply as

\[
x' = \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} x.
\]

**Real Eigenvalues**

In the calculations and expressions that follow both the matrix form (above) and the component form will be given and separated by “or.” They are equivalent. Based on our discussion in class, we expect a solution of (1) to have the form

\[
x = \begin{pmatrix}
  \xi_1 \\
  \xi_2
\end{pmatrix} e^{rt} \quad \text{or} \quad x_1 = \xi_1 e^{rt} \quad x_2 = \xi_2 e^{rt}
\]

(3)

where as of now, \(\xi_1, \xi_2,\) and \(r\) are unknown. To be more concrete, we consider a specific system.
Example 1. Consider the system of differential equations

\[ \mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} \quad \text{or} \quad \begin{array}{l} x_1' = x_1 + x_2 \\ x_2' = 4x_1 + x_2 \end{array}, \tag{4} \]

Our guess, (3), is substituted in (4), and we find

\[ re^{rt} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{rt} \quad \text{or} \quad r\xi_1 = \xi_1 + \xi_2 \]

\[ r\xi_2 = 4\xi_1 + \xi_2. \]

Rearranging gives

\[ \begin{pmatrix} 1 - r & 1 \\ 4 & 1 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{array}{l} (1 - r)\xi_1 + \xi_2 = 0 \\ 4\xi_1 + (1 - r)\xi_2 = 0 \end{array}. \tag{5} \]

Geometrically the last system of equations describe two lines passing through the origin. We do not want a unique solution (zero in this case), for that would say \( \xi_1 = \xi_2 = 0 \), and our guess would not produce anything interesting. We want the lines in (5) to be the same, so we require the slopes to be equal. That is,

\[ r^2 - 2r - 3 = 0. \]

The roots, called *eigenvalues*, are \( r = 3, -1 \).

Left to find are \( \xi_1 \) and \( \xi_2 \). Inserting \( r = 3 \) back in (5), we find

\[ -2\xi_1 + \xi_2 = 0 \]
\[ 4\xi_1 - 2\xi_2 = 0. \]

As expected this system has infinitely many solutions, one of them is

\[ \xi = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{or} \quad \begin{array}{l} \xi_1 = 1 \\ \xi_2 = 2 \end{array}, \]

and, a solution is

\[ \mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} \quad \text{or} \quad \begin{array}{l} x_1 = e^{3t} \\ x_2 = 2e^{3t} \end{array}. \]

A similar calculation for \( r = -1 \) shows

\[ \xi = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{or} \quad \begin{array}{l} \xi_1 = 1 \\ \xi_2 = -2 \end{array}, \]

and, a solution is

\[ \mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \quad \text{or} \quad \begin{array}{l} x_1 = e^{-t} \\ x_2 = 2e^{-t} \end{array}. \]

The general solution is therefore,

\[ \mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \quad \text{or} \quad \begin{array}{l} x_1 = c_1 e^{3t} + c_2 e^{-t} \\ x_2 = c_1 2e^{3t} - c_2 2e^{-t} \end{array}. \]
Example 2. Find the general solution of the ODE
\[ y'' - 2y' - 3y = 0, \quad y(0) = 2, \quad y'(0) = 2. \]

using the approach in Example 1.

Solution. We may turn the ODE into a system of ODEs by setting \( x_1 = y \) and \( x_2 = y' \). Then the second-order ODE is equivalent to
\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
= \begin{pmatrix}
  0 & 1 \\
  3 & 2
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

A comparison of (4) and (5) in Example 1 shows that we can immediately find the equations for \( \xi \) in our guess. Here
\[
\begin{pmatrix}
  0 - r & 1 \\
  3 & 2 - r
\end{pmatrix}
\begin{pmatrix}
  \xi_1 \\
  \xi_2
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad -r\xi_1 + \xi_2 = 0 \quad 3\xi_1 + (2 - r)\xi_2 = 0.
\]

The polynomial for \( r \) is just the determinate or the upper-left diagonal element times the lower-right minus the product of the remaining two diagonal elements. Here, we find
\[
r^2 - 2r - 3 = 0.
\]
Note that this is the characteristic polynomial for the original second-order ODE! The roots are \( r = 3, -1 \).

Inserting \( r = 3 \) in the equations for \( \xi \), we find
\[
-3\xi_1 + \xi_2 = 0 \\
3\xi_1 - \xi_2 = 0.
\]

As expected this system has infinitely many solutions, one of them is
\[
\xi = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{or} \quad \xi_1 = 1 \\
\xi_2 = 3.
\]

The other root, \( r = -1 \) gives
\[
\xi_1 + \xi_2 = 0 \\
3\xi_1 + 3\xi_1 = 0,
\]

and
\[
\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{or} \quad \xi_1 = 1 \\
\xi_2 = -1.
\]

The general solution is therefore
\[
x = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \quad \text{or} \quad x_1 = c_1 e^{3t} + c_2 e^{-t} \\
x_2 = c_1 3e^{3t} - c_2 e^{-t}.
\]
Since we set \( y = x_1 \) in the original transformation, the top line is the general solution to the second-order ODE, and the bottom line is the derivative of the top (\( x_2 = y' = x_1' \)).

To find \( c_1 \) and \( c_2 \) we use the initial data -
\[
\begin{pmatrix} 2 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]
Solving, we find \( c_1 = 1 = c_2 \), and the solution is
\[
x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \quad \text{or} \quad y = e^{3t} + e^{-t} \\
y' = 3e^{3t} - e^{-t}.
\]
In this section we mostly drop the component form of the equations (get used to it). As in previous problems we use Euler’s formula to change exponents with complex numbers to oscillatory functions.

If \( z = a + ib \) is a complex number, the complex conjugate is \( \bar{z} = a - ib \). That is, \( i \) is replaced with \( -i \). In addition, the real and imaginary parts are denoted

\[
\text{Re}(z) = a \quad \text{Im}(z) = b.
\]

Both are real numbers.

**Example 3** Solve the initial-value problem

\[
x' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

**Solution.** As in the previous example, the system for \( \xi \) is

\[
\begin{pmatrix} 2 - r & 8 \\ -1 & -2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

The nontrivial solution is obtained by requiring the determinant to be zero. That is,

\[
(2 - r)(-2 - r) - (-1)(8) = r^2 + 4 = 0.
\]

The eigenvalues (roots) are \( r = \pm 2i \). The procedure is the same as in Example 2. Suppose \( r = 2i \). The equations for \( \xi \) are

\[
\begin{align*}
(2 - 2i)\xi_1 + 8\xi_2 &= 0 \\
-1\xi_1 + (-2 - 2i)\xi_2 &= 0.
\end{align*}
\]

It is not obvious the two equations are multiples of one another. However, you can verify that \((-2 + 2i)\) times the second equation gives the first equation. A non-trivial solution is required. If we opportunistically set \( \xi_2 = 1 \) in the second equation, then \( \xi_1 = -2 - 2i \). So

\[
\xi = \begin{pmatrix} 2 + 2i \\ -1 \end{pmatrix}
\]

will work, and one solution is

\[
x^{(1)} = \begin{pmatrix} 2 + 2i \\ -1 \end{pmatrix} e^{2it}.
\]

Since the other root (eigenvalue) is just the complex conjugate of the first, a second solution is found simply by changing \( i \) in the first solution to \( -i \) (its complex conjugate). So the general solutions is

\[
x = C_1 \begin{pmatrix} 2 + 2i \\ -1 \end{pmatrix} e^{2it} + C_2 \begin{pmatrix} 2 - 2i \\ -1 \end{pmatrix} e^{-2it}.
\]

The complex numbers are undesirable. We hide them by using Euler’s formula: \( e^{i\theta} = \cos \theta + i \sin \theta \). The exponential functions in (6) are replaced, and the result expressed as the real plus imaginary part -

\[
x = C_1 \begin{pmatrix} 2 + 2i \\ -1 \end{pmatrix} (\cos 2t + i \sin 2t) + C_2 \begin{pmatrix} 2 - 2i \\ -1 \end{pmatrix} (\cos 2t - i \sin 2t).
\]
Read across the first line and pick out all the terms without an $i$. Then do the same for the bottom line (that is, find the real part). The result is

$$\text{Re}(x) = (C_1 + C_2) \begin{pmatrix} 2 \cos 2t - 2 \sin 2t \\ -\cos 2t \end{pmatrix}.$$  

The imaginary part consists of all the terms with an $i$ in front. In particular,

$$i\text{Im}(x) = i(C_1 - C_2) \begin{pmatrix} 2 \cos 2t + 2 \sin 2t \\ -\sin 2t \end{pmatrix}.$$  

Finally, set $c_1 = C_1 + C_2$ and $c_2 = i(C_1 - C_2)$. Then

$$x = c_1 \begin{pmatrix} 2 \cos 2t - 2 \sin 2t \\ -\cos 2t \end{pmatrix} + c_2 \begin{pmatrix} 2 \cos 2t + 2 \sin 2t \\ -\sin 2t \end{pmatrix}$$  

is the general solution.

There is a slight short-cut to this procedure. Since there is so much symmetry in the solutions for the two roots (they differ by replacing $i$ with $-i$), one might expect that all the necessary information is contained in one of the solutions. This is the case. The solution for $r = 2i$ (first part of Equation (6)) is

$$x^{(1)} = \begin{pmatrix} 2 + 2i \\ -1 \end{pmatrix} e^{2it} = \begin{pmatrix} 2 + 2i \\ -1 \end{pmatrix} (\cos 2t + i \sin 2t).$$  

To find the general solution, only the real and imaginary parts of this solution need to be found. Again rewriting as real plus imaginary,

$$x^{(1)} = \begin{pmatrix} 2 \cos 2t - 2 \sin 2t \\ -\cos 2t \end{pmatrix} + i \begin{pmatrix} 2 \cos 2t + 2 \sin 2t \\ -\sin 2t \end{pmatrix}.$$  

The general solution is $x^{(1)} = c_1 \text{Re}(x^{(1)}) + c_2 \text{Im}(x^{(1)})$ or

$$x = c_1 \begin{pmatrix} 2 \cos 2t - 2 \sin 2t \\ -\cos 2t \end{pmatrix} + c_2 \begin{pmatrix} 2 \cos 2t + 2 \sin 2t \\ -\sin 2t \end{pmatrix}$$  

as before.

**Homework (Complex)**

In Problems 1-5 find the general solution, and sketch a phase portrait.

1. $x' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} x$

2. $x' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} x$

3. $x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x$

4. $x' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x$

5. $x' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} x$
Answers

1. \[ x = c_1 e^{t} \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^{t} \begin{pmatrix} \sin 2t \\ -\cos 2t + \sin 2t \end{pmatrix} \]

2. \[ x = c_1 e^{-t} \begin{pmatrix} 2 \cos 2t \\ \sin 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -2 \sin 2t \\ \cos 2t \end{pmatrix} \]

3. \[ x = c_1 \begin{pmatrix} 5 \cos t \\ 2 \cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5 \sin t \\ -\cos t + 2 \sin t \end{pmatrix} \]

4. \[ x = c_1 e^{-t} \begin{pmatrix} \cos t \\ 2 \cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin t \\ -\cos t + 2 \sin t \end{pmatrix} \]

5. \[ x = c_1 \begin{pmatrix} -2 \cos 3t \\ \cos 3t + 3 \sin 3t \end{pmatrix} + c_2 \begin{pmatrix} -2 \sin 3t \\ -3 \cos 3t + \sin 3t \end{pmatrix} \]

Repeated Eigenvalues

Of course the eigenvalues need not all be distinct. Consider for example

\[ x' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} x. \] (7)

Proceeding as before, the system for \( \xi \) is

\[ \begin{pmatrix} 1 - r & -1 \\ 1 & 3 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \] (8)

The nontrivial solution is obtained by requiring the determinate to be zero. That is,

\( (1 - r)(3 - r) - (-1) = r^2 - 4r + 4 = 0. \)

The eigenvalues (roots) are \( r = 2, 2. \) The equations for \( \xi \) are

\[-\xi_1 - \xi_2 = 0 \]
\[ \xi_1 + \xi_2 = 0. \]

A nontrivial solution is

\[ \xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \]

and one solution is

\[ x^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}. \]

In analogy with second-order, linear, constant coefficient, homogeneous ODEs, we might expect the second solution to be

\[ x^{(2)} = tx^{(1)} = t \left( \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} \right). \] (9)
Unfortunately, this is NOT a solution. What to do?

Somehow a candidate for the second solution has to be constructed from the only solution we have. Based on previous experiences, (9) has to be close to the correct solution. We alter it slightly and try again. Set

\[ x^{(2)} = tx^{(1)} + \eta e^{2t} = t \left(\begin{array}{c} 1 \\ -1 \end{array}\right) e^{2t} + \left(\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right) e^{2t}. \]  

(10)

We have to see if a choice for \( \eta \) exists so that \( x^{(2)} \) solves (7). The left side of (7) (the time derivative of (10)) is

\[ x^{(2)'} = x^{(1)'} + \left(\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right) 2e^{2t} = \left(\begin{array}{c} 1 \\ -1 \end{array}\right) e^{2t} + 2t \left(\begin{array}{c} 1 \\ -1 \end{array}\right) e^{2t} + \left(\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right) 2e^{2t}. \]  

(11)

The right side is

\[ \left(\begin{array}{cc} 1 & -1 \\ 1 & 3 \end{array}\right) \left(\begin{array}{c} 1 \\ -1 \end{array}\right) te^{2t} + \left(\begin{array}{cc} 1 & -1 \\ 1 & 3 \end{array}\right) \left(\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right) e^{2t}. \]  

(12)

However,

\[ \left(\begin{array}{cc} 1 & -1 \\ 1 & 3 \end{array}\right) \left(\begin{array}{c} 1 \\ -1 \end{array}\right) te^{2t} = 2t \left(\begin{array}{c} 1 \\ -1 \end{array}\right) e^{2t}. \]

Equating (11) and (12), canceling the common \( e^{2t} \) and the other common terms, we find

\[ \left(\begin{array}{c} 1 \\ -1 \end{array}\right) + \left(\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right) 2 = \left(\begin{array}{cc} 1 & -1 \\ 1 & 3 \end{array}\right) \left(\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right). \]

After rearranging (\( r = 2 \) here)

\[ \left(\begin{array}{cc} 1 - r & -1 \\ 1 & 3 - r \end{array}\right) \left(\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right) = \left(\begin{array}{c} 1 \\ -1 \end{array}\right) \]

or more generally the system for \( \eta \) is

\[ \left(\begin{array}{cc} 1 - r & -1 \\ 1 & 3 - r \end{array}\right) \left(\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right) = \left(\begin{array}{c} \xi_1 \\ \xi_2 \end{array}\right). \]  

(13)

Notice (13) is an iteration of (8). For the current example, (13) in component form is

\[ -\eta_1 - \eta_2 = 1 \\
\eta_1 + \eta_2 = -1. \]

As for the system for \( \xi \), the two equations for \( \eta \) are multiples of one another. Any solution will suffice. The simplest way to find a solution is to set either \( \eta_1 \) or \( \eta_2 \) to zero. Hence, \( \eta_1 = -1 \) and \( \eta_2 = 0 \) will work, and the second solution is

\[ x^{(2)} = t \left(\begin{array}{c} 1 \\ -1 \end{array}\right) e^{2t} + \left(\begin{array}{c} -1 \\ 0 \end{array}\right) e^{2t}, \]

and the general solution is

\[ x = c_1 \left(\begin{array}{c} 1 \\ -1 \end{array}\right) e^{2t} + c_2 \left[ \left(\begin{array}{c} 1 \\ -1 \end{array}\right) te^{2t} + \left(\begin{array}{c} -1 \\ 0 \end{array}\right) e^{2t} \right]. \]
**Example 4.** Find the solution of the following system of differential equations.

\[
x' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.
\]

**Solution.** The system for \( \xi \) is

\[
\begin{pmatrix} 3 - r & 9 \\ -1 & -3 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

The nontrivial solution is obtained by requiring the determinant to be zero. That is,

\[(3 - r)(-3 - r) - (-1)(9) = r^2 = 0.\]

The eigenvalues (roots) are \( r = 0, 0 \). The equations for \( \xi \) are

\[
\begin{align*}
3\xi_1 + 9\xi_2 &= 0, \\
1\xi_1 + 3\xi_2 &= 0.
\end{align*}
\]

A nontrivial solution is \( \xi_1 = 3, \xi_2 = -1 \), and

\[
x^{(1)} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.
\]

The system for \( \eta \) is

\[
\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.
\]

Any non-trivial solution is sufficient. The choice \( \eta_2 = 0 \) and \( \eta_1 = 1 \) is a solution. Therefore, the general solution is

\[
x = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 3 + 42t \\ -4 - 14t \end{pmatrix}.
\]

To find the solution, the initial data must be applied. At \( t = 0 \)

\[
\begin{pmatrix} 2 \\ 4 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

We find \( c_1 = -4 \) and \( c_2 = 14 \), and the solution is

\[
x(t) = \begin{pmatrix} 2 + 42t \\ -4 - 14t \end{pmatrix}.
\]
Homework (Repeated)

In Problems 1-5 find the general solution, and the solution if initial data is provided. Also sketch a phase portrait.

1. \( \mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x} \)

2. \( \mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x} \)

3. \( \mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \mathbf{x} \)

4. \( \mathbf{x}' = \begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix} \mathbf{x} \)

5. \( \mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x} \) \( \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \)

Answers

1. \( \mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \left[ \begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \right] \).

2. \( \mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \right] \).

3. \( \mathbf{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \left[ \begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix} \right] \).

4. \( \mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{2t} + \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} e^{2t} \right] \).

5. \( \mathbf{x} = \begin{pmatrix} 3 + 4t \\ 2 + 4t \end{pmatrix} \).