Chapter 6

Nonlinear Systems

We consider the nonlinear system

$$x' = F(x, y),\\ y' = G(x, y), \tag{6.1}$$

where F and G are smooth functions. We assume the solution exists for all $t \geq 0$ and is unique when initial data is provided. Typically, explicit solutions to (6.1) cannot found. However, as we will see, we can construct a phase portrait for the nonlinear system without finding the solutions.

Definition 6.1. A *critical point*, (x^*, y^*), (also called an equilibrium, fixed, or stationary point) satisfies

$$F(x^*, y^*) = 0 = G(x^*, y^*).$$

Definition 6.2. A *trajectory* starting at x_0, y_0 is the set

$$\Big\{(x(t), y(t)) \ | \ t \geq 0, \quad (x(0), y(0)) = (x_0, y_0)\Big\}.$$

Here are some facts about the solutions to the nonlinear system, (6.1):

- If the initial data is a critical point, then the solution remains at the critical point for all time.
- Trajectories cannot intersect.
- In particular, trajectories not starting at a critical point can never intersect (in finite time) a critical point.
- The interesting dynamics occur near the critical points.

It turns out that we can usually figure out the behavior of the nonlinear system near a critical point. The rest of the phase portrait can usually be deduced from this information. To determine the behavior near a critical point, we will *linearize* the nonlinear system around the critical point and use our knowledge of linear systems. We hope the full nonlinear system inherits the behavior of the linearized system. As we shall see, this is frequently the case.

The Linearization

To find the linear approximation to the function $f(x)$ near a point x^*, we of course use Taylor's theorem:

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*).$$

There is a similar version of this theorem for functions depending on two variables. Indeed,

$$F(x, y) \approx F(x^*, y^*) + \frac{\partial F}{\partial x}(x^*, y^*)(x - x^*) + \frac{\partial F}{\partial y}(x^*, y^*)(y - y^*).$$

At a critical point, $F(x^*, y^*) = 0 = G(x^*, y^*)$. So the linear approximation to (6.1) is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \frac{\partial F}{\partial x}(x^*, y^*) & \frac{\partial F}{\partial y}(x^*, y^*) \\ \frac{\partial G}{\partial x}(x^*, y^*) & \frac{\partial G}{\partial y}(x^*, y^*) \end{pmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}. \tag{6.2}$$

We can clean this up a little if we set $u = x - x^*$ and $v = y - y^*$. Then the linearization near (x^*, y^*) becomes

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix}' &= \begin{pmatrix} \frac{\partial F}{\partial x}(x^*, y^*) & \frac{\partial F}{\partial y}(x^*, y^*) \\ \frac{\partial G}{\partial x}(x^*, y^*) & \frac{\partial G}{\partial y}(x^*, y^*) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &:= J \begin{pmatrix} u \\ v \end{pmatrix}. \end{aligned}$$

The matrix J is called the Jacobian.

Before stating the theorems relating the behavior of the linearized system to the full nonlinear system, we need first summarize the behavior of linear systems. We observe three types of behaviors in linear systems. They are (note the origin is always a critical point in a linear system)

- all solutions converge to the origin as $t \to \infty$. This happens when the eigenvalues are negative or the have negative real part. In this case we call the origin *asymptotically stable*.
- solutions near the origin stay near the origin for all time. This happens when the eigenvalues are purely complex or there is an eigenvalue which is zero while the other is negative. In this case we call the origin *stable*..
- if neither of the above two occur, we call the origin *unstable*. That is, at least one trajectory leaves the vicinity of the origin.

Here are the relevant theorems.

Theorem 6.3. *(Poincaré-Lyapunov) Suppose x^*, y^* is a critical point of the nonlinear system* (6.1)*, and suppose the $Re(\lambda)$, the real part of the eigenvalues of J (the linearization) are negative. Then the critical point is locally asymptotically stable.*

Theorem 6.4. *Suppose x^*, y^* is a critical point, and the real part of at least one eigenvalue of J is positive. Then the critical point is unstable.*

These theorems only describe the local behavior of solutions near a fixed point. They don't say what happens to the phase portrait. For that we have

Theorem 6.5. *(Grobman-Hartman) Suppose x^*, y^* is a hyperbolic critical point (*i.e. *the real part of the eigenvalues of J are not zero). Then the phase portrait of the linearization and the nonlinear equations are locally homeomorphic.*

This just says the phase portraits of the linearization and nonlinear equations are similar provided none of the eigenvalues of the linearization have zero real part.

To summarize:

Robust Cases:

- *Sources or Repellers*: both eigenvalues have positive real part.
- *Sinks or Attractors:* both eigenvalues have negative real part.
- *Saddles:* one eigenvalues is positive and the other negative.

Marginal Cases:

- *Focus or Center*: eigenvalues are pure imaginary. Linearized system does **NOT** describe the nonlinear system.
- *Zero Eigenvalue*: usually results from non isolated critical points. If the other eigenvalue is positive, the critical point is unstable. If the other eigenvalue is negative, the linearization may **NOT** describe the nonlinear system.

Example 6.6. Consider

$$\begin{aligned} x' &= y \\ y' &= x + 2x^3. \end{aligned}$$

We first find the critical points. We must solve $x' = y' = 0$. This implies $y = 0$ and $x(1 + 2x^2) = 0$. That is, $x = 0 = y$ is the only critical point.

Next we find the linearization. Here $\dfrac{\partial F}{\partial x} = 0$, $\dfrac{\partial F}{\partial y} = 1$, $\dfrac{\partial G}{\partial x} = 1 + 6x^2$, and $\dfrac{\partial G}{\partial y} = 0$. Thus

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By inspection $\Delta = -1$ and $\tau = 0$. Therefore the linearization implies a saddle. Theorem 6.4 implies $(0,0)$ is unstable. The two phase portraits are given below. Note the two are similar in accordance with Theorem 6.5.

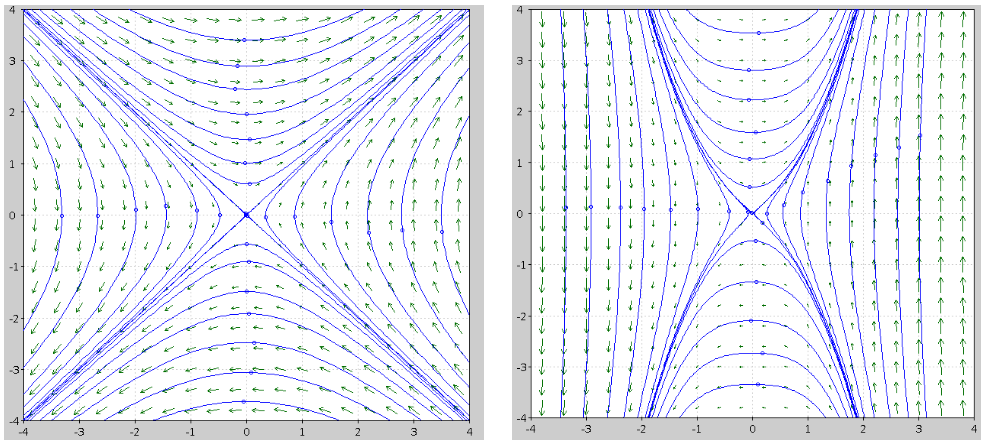

Figure 1. Phase Portrait for the linearization in Example 6.6(left), and the nonlinear equations (right).

Example 6.7. Consider

$$\begin{aligned} x' &= 2x + y + xy^3 \\ y' &= x - 2y - xy. \end{aligned}$$

We first find the critical points. We must solve $x' = y' = 0$. The equation for $\dot{y} = 0$ implies $y = \frac{x}{2+x}$. Plugging this back in the equation for $\dot{x} = 0$, we find

$$2x + \frac{x}{2+x} + \frac{x^4}{(2+x)^3} = 0,$$

or

$$x(3x^3 + 13x^2 + 28x + 20) = 0.$$

The roots are zero and

$$x^* = \frac{(287 + 18(2019)^{2/3} - 83 - 13(287 + 18\sqrt{2019}))^{1/3}}{9(287 + 18\sqrt{2019})^{1/3}}.$$

Thus the critical points are $(0, 0)$ and approximately, $(-1.1934, -1.4797)$.

The Jacobian is

$$J(x, y) = \begin{pmatrix} 2 + y^3 & 1 + 3xy^2 \\ 1 - y & -2 - x \end{pmatrix}.$$

Thus

$$J(0, 0) = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

Here $\tau = 0$, $\Delta = -5$, and the origin is a saddle. For the other critical point we find the approximate eigenvalues $r = -1.023 \pm 4.11i$, and $(-1.1934, -1.4797)$ is an asymptotically stable spiral.

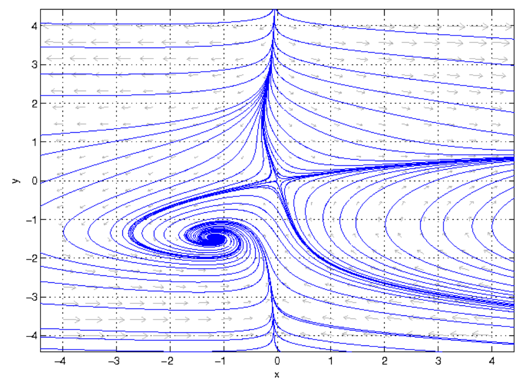

Figure 2. Phase Portrait for the linearization in Example 6.7.

Homework 6.1

In each problem find the critical points and the corresponding linear system. Find the eigenvalues of the linearized system and sketch (when possible) a local phase portrait.

1. $x' = x - y, \quad y' = x^2 + y^2 - 1$
2. $x' = 1 - y, \quad y' = x^2 - y^2$
3. $x' = x + x^2 + y^2, \quad y' = y - xy$
4. $x' = (1+x)\sin y, \quad y' = 1-x-\cos y$
5. $x' = -2x+2x^2$, $y' = -3x+y+3x^2$
6. $x' = (y-x)(1-x-y)$, $y' = x(2+y)$
7. Consider a pendulum with damping. The governing equations are

$$m\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + mgL\sin\theta = 0.$$

with $\gamma > 0$. If we set $m = 1$, $gL = 1$, $\dfrac{d\theta}{dt} = \dot{\theta}$ and $\nu = \dot{\theta}$, we find the system

$$\begin{aligned} \dot{\theta} &= \nu \\ \dot{\nu} &= -\sin\theta - \gamma\nu. \end{aligned}$$

Find all of the critical points and the associated linearize systems. Using the theorems in the chapter, sketch a phase portrait for the nonlinear system. Interpret the phase portrait physically.

8. Consider the predator-prey model given by

$$\begin{aligned}\dot{x} &= x(1 - ax - cy)\\ \dot{y} &= y(1 - by + dx).\end{aligned}$$

   a) Interpret the model and parameters biologically (what does small/large a, b, c, d mean?
   b) The fixed points are

$$(0,0), \quad (1/a, 0), \quad (0, 1/b), \quad \left(\frac{b-c}{ab+cd}, \frac{a+d}{ab+cd}\right).$$

   Check this and interpret the fixed points biologically.
   c) Linearize the system about the fixed points $(0,0)$, $(1/a, 0)$, $(0, 1/b)$ and describe the local phase portrait in the first quadrant. You will discover that the fixed point $(0, 1/b)$ has two different phase portraits depending on the sign of $b-c$. You can deduce the phase portrait near the remaining fixed point (coexistence) from other fixed points.
   d) sketch or use *pplane* (with appropriate values for a, b, c, d) to make phase portraits when $b > c$ and $b < c$.

9. Next next problem is meant to model violence after a society collapses. We let $S(t) \geq 0$ stand for susceptible and $V(t) \geq 0$ represent people who turn violent. The model is

$$\begin{aligned}\dot{S} &= S(1 - S) - SV + \frac{1}{4}V\\ \dot{V} &= SV - .25V - .25V.\end{aligned}$$

   We repeat the term $.25V$ to model a death rate for the violent class, and a return to the susceptible class for some.

   Find the steady states and sketch a phase portrait for the nonlinear system. Do the model behave as expected?

Answers

1. $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, unstable spiral; $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, saddle
2. $(1,1)$, asymptotically stable spiral; $(-1,1)$, saddle
3. $(0,0)$, unstable node; $(-1,0)$, saddle
4. $(0, 2n\pi)$, indeterminate; $(2, (2n-1)\pi)$, saddle
5. $(0,0)$, saddle; $(1,0)$, unstable node
6. $(0,0)$, saddle; $(0,1)$, asymptotically stable spiral; $(-2,-2)$, asymptotically stable node; $(3,-2)$, unstable node