First-Order ODEs

We consider in this chapter differential equations of the form

\[
\frac{dy}{dt} = F(t, y),
\]

where \( F(t, y) \) is a known smooth function. We wish to solve for \( y(t) \). Equation (1.1) is called a first-order ordinary differential equation (ODE). First order because the number of derivatives on the unknown function is one. Of course

\[
\frac{d^2y}{dt^2} = F(t, y),
\]

would be a second-order differential equation. Equation (1.1) is called an ordinary differential equation because the unknown function, \( y(t) \), depends only on one parameter - here \( t \). If (1.1) had more independent parameters and derivatives involving those variables, the differential equation would be called a partial differential equation (PDE).

One nice feature of first-order equations is that often their solutions can be found explicitly. We will see this is not the case for higher order equations. In this chapter we will only discuss two methods for finding explicit solutions to (1.1).

1.1. Separable Equations

Consider the first-order ODE

\[
y' = \frac{t^2}{y(1 + t^3)},
\]

where \( y' = \frac{dy}{dt} \). In this case the ODE may be written

\[
yy' = \frac{t^2}{1 + t^3}.
\]

It is called separable because it has the form \( G(y)y' = H(t) \), and both sides of the equation can be integrated to find an implicit equation for \( y(t) \). Indeed, integrating
both sides of (1.2), we find

\[ \frac{y^2}{2} = \frac{1}{3} \ln(1 + t^3) + c. \]  

We cannot solve for \( y \) exactly here since we do not know which root to take. If we were told that the solution must pass through \((0, 1)\), that is, \( y(0) = 1 \), then we would know the positive root is desired. Plugging in \( t = 0 \) and forcing \( y = 1 \) in (1.3), we find

\[ \frac{1}{2} = \frac{1}{3} \ln(1) + c = 0 + c, \]

and \( c = 1/2 \). Since \( y(0) \) is positive, we want the positive root and

\[ y(t) = \left( \frac{2}{3} \ln(1 + t^3) + 1 \right)^{1/2}. \]

**Example 1.1.** Solve

\[ y' = \frac{2t}{1 + 2y}, \quad y(2) = 0. \]

The trick is, if possible, to get the variables separated (\( y' \)’s on one side, \( t' \)’s on the other). Here we find

\[(1 + 2y)y' = 2t,\]

and integrating both sides

\[ y + y^2 = t^2 + c. \]

We use the data to find \( c \). Here \( y(2) = 0 \) implies

\[ 0 + 0^2 = 2^2 + c, \]

or \( c = -4 \). Solving for \( y \) we find

\[ y = \frac{-1 \pm \sqrt{1 - 4(-t^2 + 4)}}{2} = -\frac{1}{2} \pm \frac{1}{2} \sqrt{4t^2 - 15}. \]

To determine which root to pick we return to the initial data. We want \( y(2) = 0 \). This implies the positive root, and

\[ y = -\frac{1}{2} + \frac{1}{2} \sqrt{4t^2 - 15}. \]

**Homework 1.1 (Separable ODEs)**

Solve for \( y(t) \) at least implicitly. When possible solve for \( y \).

1. \( y' = y^2 \)
2. \( y' = \frac{t}{y(1 + t^2)} \)
3. \( y' = \sin^2(t) \cos(y) \)
4. \( t + ye^{-t}y' = 0, \quad y(0) = 1 \)

**Answers**

1. \( y = \frac{-1}{(t + c)} \)
2. \( y^2 - \ln(1 + t^2) = c \)
3. \( \ln |\tan y + \sec y| - t/2 - \sin(2t)/4 = c \)
4. \( y = (2(1 - t)e^t - 1)^{1/2} \)
1.2. First-Order Linear Equations

Now we consider a special case of (1.1). Specifically, we consider the initial data problem

\[\frac{dy}{dt} + p(t)y = q(t),\]

\[y(t_0) = y_0,\]

where \(p(t)\) and \(q(t)\) are given functions. Equation (1.4) is called a first-order linear ODE. We will explain the meaning of linear soon (Chapter 3). Equation (1.4) is not separable, but it is almost separable.

As a concrete example, suppose we wish to solve

\[y' + y = e^t,\]

\[y(0) = 1.\]

Here \(p = 1\) and \(q = e^t\). To find a method for solving the ODE, consider two alterations of (1.5), both of which are examples of (1.4).

**Example 1.2.** Solve

\[y' = e^t.\]

Here \(p = 0\) and \(q = e^t\) in (1.4). You could solve this ODE before taking the course. Indeed, integrating both sides gives

\[y = e^t + c.\]

**Example 1.3.** Solve

\[y' + y = 0.\]

Here \(p = 1\) and \(q = 0\) in (1.4). You could solve this ODE too before taking the course. Indeed, it is separable. Dividing by \(y\) and integrating, we find

\[\ln y = -t + c.\]

This implies

\[e^{\ln y} = y = e^{-t+c} = ce^{-t}.\]

Since we do not know \(c\), we can replace \(e^c\) with \(c\) which represents a generic constant. Thus

\[y = ce^{-t}.\]

Somehow solving (1.4) should be a combination of these two examples. Specifically, if we could make the left side of (1.4) the derivative of one quantity, as in Example 1.2, we could integrate both sides of (1.4) and find a solution. To keep the example simple, we multiply both sides of (1.5) by an unknown function \(\mu(t)\) in hopes that \(\mu\) may be chosen so that

\[\mu(y' + y) = (\mu y)'.\]

If we can do this, (1.5) becomes

\[(\mu y)' = \mu e^t,\]

and the solution can be found by integrating both sides.
We need to find the function $\mu$. By the product rule

$$(\mu y)' = \mu' y + \mu y'.$$

The original left side of (1.5) is $\mu(y' + y)$. Setting the two equal requires

$$\mu' y + \mu y' = \mu y' + \mu y,$$

or

$$y(\mu' - \mu) = 0.$$ 

Since we know nothing about $y$, and $\mu$ is at our choosing, we set $\mu' - \mu = 0$. This ODE is separable and has the general solution

$$\mu(t) = ce^t.$$ 

Again since $\mu$ may be chosen any way we want, we set $c = 1$ here, and we find that multiplying (1.5) by $e^t$ changes the ODE to

$$(ye^t)' = e^t e^t = e^{2t}$$

(check this!). We may now integrate both sides to find

$$ye^t = \frac{1}{2} e^{2t} + c$$

or

$$y = \frac{1}{2} e^t + ce^{-t}.$$ 

To find the constant we use the initial data. Here $y(0) = 1$, thus

$$1 = y(0) = \frac{1}{2} e^0 + ce^0 = \frac{1}{2} + c,$$

and $c = 1/2$. Therefore the solution to

$$y' + y = e^t,$$

$$y(0) = 1.$$ 

is

$$y = \frac{1}{2} e^t + \frac{1}{2} e^{-t}$$

(check this!!).

More generally multiplying (1.4) by $\mu(t) = e^{\int p \, dt}$

changes (1.4) to

$$(y\mu)' = q(t)\mu(t)$$

which can be solved by integrating both sides. The function $\mu$ is called an integrating factor.

**Example 1.4.** Find the general solution to

$$y' - 2y = t^2 e^{2t}.$$ 

Here $p = -2$, and so

$$\mu(t) = e^{\int -2 \, dt} = e^{-2t}.$$
The original ODE becomes

\[(ye^{-2t})' = t^2e^{2t}e^{-2t} = t^2.\]

Integrating both sides

\[ye^{-2t} = \frac{1}{3}t^3 + c,
\]
or

\[y = \frac{1}{3}t^3e^{2t} + ce^{2t}.
\]

**Example 1.5.** Find the solution to

\[ty' + y = \cos t\]

\[y(\pi/2) = 2/\pi.\]

Not that this ODE is not in the form given in (1.4). We must divide by \(t\) first.

We can make a procedure:

**Step 0.** Put the ODE in the form given in (1.4).

\[(1.6)\]

\[y' + \frac{1}{t}y = \cos t\]

**Step 1.** Find \(\mu\). Here

\[\mu = e^{\int \frac{1}{t} \, dt} = e^{\ln t} = t.
\]

**Step 2.** Multiply (1.6) in Step 0 by \(\mu\) and \textbf{NOT} the original ODE. Here

\[(yt)' = \cos t\]

(which happens to be the original ODE, but we did not know at first that the left side of ODE was already the derivative of a product).

**Step 3.** Integrate both sides. Here

\[ty = \sin t + c,
\]
or

\[y = \frac{\sin t}{t} + \frac{c}{t}.
\]

To find the constant we use the initial data. Here \(y(\pi/2) = 2/\pi\). Thus

\[\frac{2}{\pi} = y(\pi/2) = \frac{\sin(\pi/2)}{\pi/2} + \frac{c}{\pi/2},
\]

and \(c = 0\). The solution is \(y = \sin(t)/t\).

**Example 1.6.** Find the general solution to

\[ty' - 2y = t^4e^t.
\]

**Step 0.** Put the equation in the correct form:

\[y' - \frac{2}{t}y = t^3e^t.
\]

**Step 1.** Find \(\mu\):

\[\mu = e^{\int -\frac{2}{t} \, dt} = e^{-2\ln t} = \frac{1}{t^2}.
\]
Step 2. Multiply the equation in Step 0 by $\mu$:

$$
\left( \frac{y}{t^2} \right)' = te^t.
$$

Step 3. Integrate!

$$
\frac{y}{t^2} = te^t - e^t + c,
$$

and the general solution is

$$
y = t^3e^t - t^2e^t + ct^2.
$$

Homework 1.2 (Linear First Order ODEs)

Solve for $y(t)$.

1. $y' - y = te^{2t}, \quad y(0) = 1$
2. $y' - y = te^t, \quad y(0) = 1$
3. $ty' + 2y = t^2 + t + 1, \quad y(1) = 7/12, \quad t > 0$
4. $y' + \frac{3}{t} y = \frac{\sin t}{t^3}, \quad y(\pi) = 0, \quad t > 0$
5. $ty' + 2y = \sin t, \quad y(\pi/2) = 1, \quad t > 0$
6. $t^3y' + 4t^2y = e^{-t}, \quad y(-1) = 0, \quad t < 0$
7. $ty' + (t + 1)y = t, \quad y(\ln 2) = 1, \quad t > 0$

Answers

1. $y(t) = 2e^t + (t - 1)e^{2t}$
2. $y(t) = (t^2/2 + 1)e^t/2$
3. $y(t) = t^2/4 + t/3 + 1/t - 1/t^2$
4. $y(t) = -\cos t/t^5$
5. $y(t) = \frac{1}{t^2} \left( \frac{\pi^2}{4} - 1 - t \cos t + \sin t \right)$
6. $y(t) = -(t + 1)e^{-t}/t^4$
7. $y(t) = (t - 1 + 2e^{-1})/t$

1.3. Some Theory

The nice feature of linear first-order ODEs is that their solutions can be found explicitly. We will see later that ODEs with higher-order derivatives rarely have explicit solutions. Thus the procedure in the previous section is worth examining more carefully.

We expect solutions to ODEs originating from physics or engineering to be unique - there should be exactly one solution solving the ODE with the given initial data. We must restrict the domain of the solutions we found in the previous section for them to be unique. Indeed, consider Problem 1.2.4 from the previous section.
Example 1.7. Solve
\[ y' + \frac{2}{t}y = \frac{\cos t}{t^2}, \]
\[ y(\pi) = 0. \]

We apply our procedure

**Step 1.** Find \( \mu \). Here
\[ \mu = e^{\int \frac{2}{t} dt} = e^{2 \ln t} = t^2. \]

**Step 2.** Multiply by \( \mu \). Here
\[ (t^2 y)' = \cos t. \]

**Step 3.** Integrate both sides, and solve for \( y \):
\[ y(t) = \frac{\sin t}{t^2} + \frac{c}{t^2}. \]

To find \( c \) we require \( y(\pi) = 0 \), and we find \( c = 0 \).

Now we show the solution is not unique unless the domain is restricted in some way. Given that the solution is
\[ y(t) = \frac{\sin t}{t^2}, \]
we might be tempted to say the domain is \( (-\infty, 0) \cup (0, \infty) \) or all real number with \( t \neq 0 \). If we did this, both of the following would be solution to the ODE
\[ y(t) = \begin{cases} \frac{\sin t}{t^2} + \frac{1}{t^2} & t < 0 \\ \frac{\sin t}{t^2} & t > 0 \end{cases} \quad \text{and} \quad y(t) = \begin{cases} \frac{\sin t}{t^2} + \frac{2}{t^2} & t < 0 \\ \frac{\sin t}{t^2} & t > 0 \end{cases}. \]

This is because the free constant does not see the initial data at \( t = \pi > 0 \) for negative \( t \). If we want the solution unique, we must restrict the domain to an interval containing the point where the initial data is imposed.

We can summarize the properties of first-order linear ODEs in the following theorem.

**Theorem 1.8.** Consider the initial data problem
\[ \frac{dy}{dt} + p(t)y = q(t), \]
\[ y(t_0) = y_0. \]

Suppose \( p(t), q(t) \) are continuous on an interval, \( I \), with \( t_0 \in I \). Then the solution of (1.7) exists on all \( t \in I \) and it is unique there.

**Proof.** We just need to justify the steps in our procedure and check the uniqueness. Step 1 involves finding an anti derivative of \( p(t) \). By the fundamental theorem of Calculus we know an anti derivative exists if \( p \) is continuous. Similarly, Step 3 requires integrating \( \mu(t)q(t) \). Since \( p \) is continuous, \( \mu \) is too, and thus \( \mu(t)q(t) \) is continuous and may be integrated to find \( y(t) \). Therefore the \( y \) found in Step 3 is continuous on \( I \). Applying the initial data allows the integration constant to be found.
We need to show that the solution is unique on $I$. To do this suppose $y_1$ and $y_2$ both solve (1.7) on $I$. Set $w(t) = y_1(t) - y_2(t)$. Then $w$ solves
\[ \frac{dw}{dt} + p(t)w = 0, \]
and
\[ w(t_0) = 0. \]
Multiplying by $\mu$, we find
\[ (\mu w)' = 0, \quad w(t_0) = 0. \]
Integrating from $t_0$ to $t \in I$ gives
\[ \mu(t)w(t) - \mu(t_0)w(t_0) = 0, \]
for all $t \in I$. Since $\mu \neq 0$ for all $t$, we find $w(t) = 0$ for $t \in I$. That is $y_1 = y_2$ on $I$, and the solution is unique on $I$.

\[ \square \]

**Homework 1.3**

Find an interval on which the solution of the following ODEs exists and is unique. You need not actually find the solution.

1. $(t - 3)y' - (\ln t)y = t, \quad y(1) = 1$
2. $(\ln t)y' + 2y = \tan t, \quad y(\pi/4) = 1$
3. $(9 - t^2)y' + ty = t^2 - t + 1, \quad y(1) = 1$
4. $t(t - 4)y' + ty = 1, \quad y(2) = 0$

**Answers**

1. $0 < t < 3$
2. $1 < t < \pi/2$
3. $-3 < t < 3$
4. $0 < t < 4$