1. (25 pts) Consider the system of differential equations given by

\[
\begin{align*}
    x_1' &= -3x_1 + x_2 \\
    x_2' &= x_1 - 3x_2.
\end{align*}
\]

a) Find the general solution:

\[
x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-4t}.
\]

b) Sketch the phase portrait: See Figure 7.5.4 on p.395 in your book for a similar phase portrait.

c) Verify your solution: From part (a) \( x_1 = c_1 e^{-2t} - c_2 e^{-4t} \) and \( x_2 = c_1 e^{-2t} + c_2 e^{-4t} \). You have to check \( x_1' = -3x_1 + x_2 \) and \( x_2' = x_1 - 3x_2 \). Just plug in and check!

2. (25 pts) Consider the system of differential equations given by

\[
x' = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

a) Find the solution: The roots are \( r = -2, -2 \). After simplifying,

\[
x(t) = \begin{pmatrix} 1 + 12t \\ 1 - 4t \end{pmatrix} e^{-2t}.
\]

b) Sketch the phase portrait of the general solution. See Figure 7.8.2 on p.425 for a similar phase portrait (only reverse the arrows).

3. (25 pts) Consider the system of differential equations given by

\[
x' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]

a) Find the solution: The roots are \( r = -1 \pm 2i \). The eigenvector is \( \xi = \begin{pmatrix} 2 \\ -i \end{pmatrix} \). The solution is

\[
x(t) = \begin{pmatrix} 2 \cos(2t) - 2 \sin(2t) \\ \sin(2t) - \cos(2t) \end{pmatrix} e^{-t}.
\]

b) Sketch the phase portrait of the general solution. See Figure 7.6.2 on p.404 for a similar phase portrait.
4. (25 pts) Consider the nonlinear system

\[
x' = x - 1 \\
y' = x^2 - y^2.
\]

a) Find all critical points: \( x - 1 = 0 \) implies \( x = 1 \), and \( x^2 - y^2 = 0 \) implies \( y^2 = 1 \), or \( y = \pm 1 \). The critical points are \((1, 1)\) and \((1, -1)\).

b) Find the corresponding linear system near each critical point: For \((1, 1)\), set \( u = x - 1 \) and \( v = y - 1 \). Then

\[
\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}
\]

The roots are \( r = 1, -2 \) and \((1, 1)\) is an unstable saddle.

For \((1, -1)\), set \( u = x - 1 \) and \( v = y + 1 \). Then

\[
\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}
\]

The roots are \( r = 1, 2 \) and \((1, -1)\) is an unstable critical point (a source).