1. (20 pts) Find the homogeneous, and a suitable form for the particular solution

\[ y'''' + 4y'' = (t + 3)(t + 1) + (t + 1)\sin(2t) \]

**Solution:** The characteristic equation is \( r^2(r^2 + 4) = 0 \) and has roots \( r = 0, 0, \pm 2i \). The homogeneous solution is \( y_h = c_1 + c_2t + c_3\cos(2t) + c_4\sin(2t) \).

Step one of the algorithm for finding \( y_p \) is \( \{1, t, t^2, \sin(2t), t\sin(2t)\} \). Step four is \( \{t^3, t^4, t^2\sin(2t), t\sin(2t), t^2\cos(2t), t\cos(2t)\} \). And step five gives \( \{t^3, t^4, t^2\sin(2t), t\sin(2t), t^2\cos(2t), t\cos(2t)\} \). An appropriate guess for \( y_p \) is \( y_p = At^4 + Bt^3 + Ct^2 + Dt^2\sin(2t) + Et\sin(2t) + Ft^2\cos(2t) + Gt\cos(2t) \).

2. (10 pts) Consider the ODE

\[ y'' + ay' + y = 0. \]

Find the choices of \( a \) which will guarantee the solution will Not oscillate.

**Solution:** We don’t want complex roots. So \( b^2 - 4ac = \alpha^2 - 4 \geq 0 \). That is, \( |\alpha| \geq 2 \).

3. (20 pts) Compute the Laplace transform of

\[ f(t) = \begin{cases} 0 & 0 \leq t < 1, \\ t - 1 & 1 \leq t < 2, \\ 3 - t & 2 \leq t < 3, \\ 0 & t \geq 3. \end{cases} \]

**Solution:** Using Heaviside functions, \( f(t) = u_1(t)(t - 1) - u_2(t)(t - 1) + u_2(t)(3 - t) - u_3(t)(3 - t) \).

Using the home-cooked formula \( \mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t + c)) \), we find \( \mathcal{L}(f(t)) = e^{-s}\mathcal{L}(t) - e^{-2s}\mathcal{L}(t + 1) + e^{-2s}\mathcal{L}(1 - t) - e^{-3s}\mathcal{L}(-t) \). That is,

\[
\mathcal{L}(f(t)) = \frac{e^{-s}}{s^2} - 2\frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}.
\]

4. (20 pts, 10 each) Find the inverse Laplace transform of

(a) \( \frac{(s - 2)e^{-s}}{s^2 + 3s + 2} \) \hspace{1cm} (b) \( \frac{2s - 4}{s^2 + 2s + 10} \).

**Solution (a):** The denominator factors and

\[
\frac{s - 2}{(s + 1)(s + 2)} = \frac{-3}{s + 1} + \frac{4}{s + 2}.
\]
So

\[ L^{-1}(a) = u_1(t) \left( -3e^{t-1} + 4e^{-2(t-1)} \right). \]

**Solution (b):** The denominator does not factor, and

\[ \frac{2s - 4}{s^2 + 2s + 10} = \frac{2(s + 1)}{(s + 1)^2 + 3^2} - \frac{6}{3} \frac{3}{(s + 1)^2 + 3^2}. \]

Hence,

\[ L^{-1}(b) = 2e^{-t} \cos(3t) - 2e^{-t} \sin(3t). \]

5. (20 pts) Find the solution to

\[ y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 1, \]

where \( f(t) \) is given in Problem 3.

**Solution:** Taking the Laplace transform of both sides of the ODE,

\[ L(y) = \frac{1}{s^2 + 1} + \frac{L(f(t))}{s^2 + 1}. \]

The last term requires the partial fraction

\[ F_1(s) = \frac{1}{s^2(s^2 + 1)} = \frac{1}{s^2} - \frac{1}{s^2 + 1}. \]

Inverting it

\[ f_1(t) = t - \sin t. \]

So,

\[ L(y) = \frac{1}{s^2 + 1} + e^{-s}F_1(s) - 2e^{-2s}F_1(s) + e^{-3s}F_1(s), \]

and

\[ y = \sin t + u_1(t)f_1(t-1) - 2u_2(t)f_1(t-2) - u_3(t)f_1(t-3) \]

\[ = \sin t + u_1(t)(1 - \sin(t-1)) - 2u_2(t)(t-2 - \sin(t-2)) - u_3(t)(t-3 - \sin(t-3)) \]

6. (10 pts) Find the general solution to

\[ y'' + y = \delta(t-1). \]

(Yes, yes, no initial data is given. You know the solution has the form \( y = y_h + y_p \). Figure out \( y_p \).)

**Solution:** The homogeneous solution is \( y_h = c_1 \cos t + c_2 \sin t \). To find \( y_p \) make up any initial data and use Laplace transforms. Setting \( y(0) = y'(0) = 0 \), find \( y_p = u_1(t) \sin(t-1) \), and \( y = y_h + y_p \).