1. (15 pts) Find the general solution of the differential equation

\[ y'' - 4y' + 8y = 0. \]

**Solution:** The characteristic polynomial \( r^2 - 4r + 8 = 0 \) has roots \( r = 2 \pm 2i \). So the general solution is

\[ y = c_1 e^{2t} \cos 2t + c_2 e^{2t} \sin 2t. \]

2. (18 pts) Find the solution to the initial value problem, **AND** state the largest interval for which the solution is valid.

\[ ty' - y = t^3 e^t, \quad y(1) = 1. \]

**Solution:** The ODE implies \( \mu = \exp \left( -\int \frac{1}{t} dt \right) = e^{-\ln t} = \frac{1}{t} \). Multiplying by \( \mu \)

\[ \left( \frac{y}{t} \right)' = te^t. \]

Integrating by parts shows, \( y = t(te^t - e^t + c) \). Finally, \( y(1) = c = 1 \). So the solution is \( y = t^2 e^t - te^t + t \) on \(-\infty < t < \infty\).

3. (20 pts) Find the solution to the ODE (evaluate all constants)

\[ y'' + y' - 2y = -4t, \quad y(0) = 0, \quad y'(0) = 1. \]

**Solution:** The characteristic polynomial is \( (r - 1)(r + 2) = 0 \). So \( y_h = c_1 e^t + c_2 e^{-2t} \). Following the algorithm for determining \( y_p \),

1. \( \{t\} \)
2. \( \{1\} \)
3. \( \{1, t\} \)
4. Closed
5. Nothing in homogeneous list
6. \( y_p = At + B. \)
Plugging $y_p$ into the ODE, we find
\[ 1(A - 2B) + t(-2A) = -4t. \]
That is, $A = 2$ and $B = 1$. So $y = c_1 e^t + c_2 e^{-2t} + 2t + 1$. The initial data implies
\[
\begin{align*}
0 &= y(0) = c_1 + c_2 + 1 \\
1 &= y'(0) = c_1 - 2c_2 + 2
\end{align*}
\]
That is, $c_1 = -1$ and $c_2 = 0$. The solution is $y = -e^t + 2t + 1$.

4. (12 pts total)
(a) VERIFY $y_1 = t^2$ and $y_2 = t^{-1}$ solve the ODE $t^2 y'' - 2y = 0$ for $t > 0$.
Solution: $y_1 = t^2$, $y_1' = 2t$, and $y_1'' = 2$. Now, just check: $t^2 y_1'' - 2y_1 = t^2 2 - 2t^2 = 0$ as claimed.
The verification of $y_2$ works the same way.

(b) Can all solutions be written in the form $y = c_1 y_1 + c_2 y_2$?
Solution: Yes, the Wronskian is $W(y_1, y_2) = -3 \neq 0$. So $y_1$ and $y_2$ are not multiples of each other.

(c) Notice $y_1 = e^t$ and $y_2 = e^{-t}$ both solve the ODE $y'' - y = 0$. Can all solutions be written $y = c_1 y_1 + c_2 y_2$ with $c_1, c_2$ numbers? Why or why not?
Solution: No. In this case the Wronskian is $W(y_1, y_2) = 0$. Also, notice $y_1 = y_2 e$. The two solution $y_1, y_2$ have to be independent for the general solution to be $y = c_1 y_1 + c_2 y_2$.

5. (15 pts) Find the a suitable form for the particular solution to the initial-value problem
\[ y'' - 2y' + y = (t + 1) \sin(t) + 6e^{-t} + 7. \]
You need not evaluate any of the constants.
Solution: The characteristic polynomial is $(r - 1)(r - 1) = 0$. So $y_h = c_1 e^t + c_2 te^t$. Following the algorithm for determining $y_p$,
\[
\begin{align*}
1. & \{t \sin t, \sin t, e^{-t}, 1\} \\
2. & \{t \sin t, \sin t, t \cos t, -e^{-t}, 0\} \\
3. & \{t \sin t, \sin t, t \cos t, e^{-t}, 1\} \\
4. & \text{Repeating until closed } \{t \sin t, \sin t, t \cos t, \cos t, e^{-t}, 1\} \\
5. & \text{Nothing in homogeneous list} \\
6. & y_p = At \sin t + B \sin t + C t \cos t + D \cos t + E e^{-t} + F.
\end{align*}
\]

6. (18 pts) Find the general solution to (evaluate all constants in $y_p$)
\[ y'' + 2y' = 3 + 4 \sin(2t) \]
Solution: This was off an old quiz. See your notes. The answer is
\[ y = c_1 + c_2 e^{-2t} + \frac{3}{2} t - \frac{1}{2} \sin 2t - \frac{1}{2} \cos 2t. \]