Trigonometric Substitution

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<th>Expression</th>
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<th>Identity</th>
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<td>$\sqrt{a^2 - x^2}$</td>
<td>$x = a \sin \theta$</td>
<td>$1 - \sin^2 \theta = \cos^2 \theta$</td>
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<tr>
<td>$\sqrt{a^2 + x^2}$</td>
<td>$x = a \tan \theta$</td>
<td>$1 + \tan^2 \theta = \sec^2 \theta$</td>
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<tr>
<td>$\sqrt{x^2 - a^2}$</td>
<td>$x = a \sec \theta$</td>
<td>$\sec^2 \theta - 1 = \tan^2 \theta$</td>
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Example 1: Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2} \, dx$.

SOLUTION: Let $x = 3 \sin \theta$, where $-\pi/2 \leq \theta \leq \pi/2$. Then $dx = 3 \cos \theta \, d\theta$ and $\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2 \theta} = \sqrt{9\cos^2 \theta} = 3|\cos \theta| = 3 \cos \theta$. (Note $\cos \theta \geq 0$ because $-\pi/2 \leq \theta \leq \pi/2$.) Thus,

$$
\int \frac{\sqrt{9 - x^2}}{x^2} \, dx = \int \frac{3 \cos \theta}{9 \sin^2 \theta} \cdot 3 \cos \theta \, dx \\
= \int \frac{\cos^2 \theta}{\sin^2 \theta} \, dx \\
= \int \cot^2 \theta \, dx \\
= \int (\csc^2 \theta - 1) \, dx \\
= -\csc \theta - \theta + C.
$$

Since this is an indefinite integral, we must return to the original variable $x$. This can be done either by trigonometric identities to express $\cot \theta$ in terms of $\sin \theta = x/3$ or by drawing a diagram with $\theta$ interpreted as an angle of a right triangle. Since $\sin \theta = x/3$, we label the opposite side and the hypotenuse as having lengths $x$ and 3. Then the Pythagorean Theorem gives the length of the adjacent side as $\sqrt{9 - x^2}$, so we can simply read the value of $\cot \theta$ from the triangle, and

$$
\cot \theta = \frac{\sqrt{9 - x^2}}{x}.
$$

Since $\sin \theta = x/3$, we have $\theta = \sin^{-1}(x/3)$, and so

$$
\int \frac{\sqrt{9 - x^2}}{x^2} \, dx = -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C.
$$
Example 2: Find the area enclosed by the circle

\[ x^2 + y^2 = a^2 \]

SOLUTION: Solving the equation of the circle for \( y \), we get

\[ y^2 = a^2 - x^2 \text{ or } y = \pm \sqrt{a^2 - x^2}. \]

Because the circle is symmetric with respect to both axes, the total area \( A \) is four times the area in the first quadrant. The part of the circle in the first quadrant is given by the function

\[ y = \sqrt{a^2 - x^2}, \quad 0 \leq x \leq a, \]

and so

\[ \frac{1}{4} A = \int_0^a \sqrt{a^2 - x^2} \, dx. \]

To evaluate this integral we substitute \( x = \sin \theta \). Then \( dx = a \cos \theta \, d\theta \). No we change the limits of integration. We have \( x = 0 \), \( \sin \theta = 0 \), and so \( \theta = 0 \); when \( x = a \), \( \sin \theta = 1 \), so \( \theta = \pi/2 \). Also

\[ \sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = \sqrt{a^2 \cos^2 \theta} = a |\cos \theta| = a \cos \theta \]

since \( 0 \leq \theta \leq \pi/2 \). Therefore

\[ A = 4 \int_0^a \sqrt{a^2 - x^2} \, dx = 4 \int_0^{\pi/2} a \cos \theta \cdot a \cos \theta \, d\theta \]
\[ = 4a^2 \int_0^{\pi/2} \cos^2 \theta \, d\theta \]
\[ = 4a^2 \int_0^{\pi/2} \frac{1}{2} (1 + \cos 2\theta) \, d\theta \]
\[ = 2a^2 \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = 2a^2 \left[ \frac{\pi}{2} + 0 - 0 \right] \]
\[ = \pi a^2. \]

We have shown that the area of a circle is \( \pi a^2 \).

Example 3: Evaluate \( \int \frac{1}{\sqrt{x^2 - a^2}} \, dx \), where \( a > 0 \).

SOLUTION: We let \( x = a \sec \theta \) where \( 0 < \theta < \pi/2 \) or \( \pi < \theta < 3\pi/2 \). Then \( dx = a \sec \theta \tan \theta \, d\theta \), and

\[ \sqrt{x^2 - a^2} = \sqrt{a^2(\sec^2 \theta - 1)} = \sqrt{a^2 \tan^2 \theta} = a |\tan \theta| = a \tan \theta. \]

Therefore

\[ \int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \int \frac{a \sec \theta \tan \theta}{a \tan \theta} \, d\theta \]
\[ = \int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C. \]
By drawing a right triangle we find in the same way as before \( \tan \theta = \frac{\sqrt{x^2 - a^2}}{a} \). So we have

\[
\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C
\]

\[
= \ln |x + \sqrt{x^2 - a^2}| - \ln a + C.
\]

Writing \( C_1 = C - \ln a \), we have

\[
\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln |x + \sqrt{x^2 - a^2}| + C_1.
\]

Compute the following integrals using trig substitution.

1. \( \int \frac{1}{x^2\sqrt{x^2 - 9}} \, dx \)
2. \( \int_{2}^{3} \frac{1}{\sqrt{x^3\sqrt{x^2 - 1}}} \, dx \)
3. \( \int \frac{1}{x^2\sqrt{25 - x^2}} \, dx \)
4. \( \int \frac{\sqrt{x^2 - a^2}}{x^4} \, dx \)
5. \( \int \frac{1}{\sqrt{1 + x^2}} \, dx \)
6. \( \int \frac{x^2}{(a^2 - x^2)^{3/2}} \, dx \)