Numerical Methods for Conservation Laws

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LITERATURE


OUTLINE

6 lectures

Week 1: Mo 11:00-12:30, Tue 12:30-14:00 Wed 11:00-12:30

Week 2: Mo 11:00-12:30, Tue 12:30-14:00 Wed 12:30-14:00

Materials: http://math.la.asu.edu/ chris → Courses

Topics:

L1 Conservation laws and conservative formulation, Difference methods, Linear waves, Diffusion, Dispersion, CFL


L4 Higher order methods, TVD schemes.

L5 Convergence analysis, Stability and compactness.

Conservation Laws

\( U_\Omega(t) \): amount of 'mass' in \( \Omega \) at time \( t \)

\( F \): flux through the boundary \( \partial \Omega \)

\[ U_\Omega(t + \Delta t) = U_\Omega(t) - \Delta t \int_{\partial \Omega} F \cdot n \, d\sigma = U_\Omega(t) - \Delta t \int_{\Omega} \nabla_x \cdot F \, dx \]

Size of \( |\Omega| \) and \( \Delta t \to 0 \):

\[ \partial_t u + \nabla_x \cdot F = 0, \quad U_\Omega(t) = \int_{\Omega} u(x,t) \, dx \]

Examples: \( F = b(u)u \) (transport), \( F = -a(u)\nabla_x u \) (diffusion)

Wave speeds

Linear problems and plane wave solutions

\[ F(u) = bu - a\nabla_x u, \quad u(x,t) = \exp[i\xi(x - vt)] \]
\( v \): velocity, \( v\xi \): frequency

\[
\begin{align*}
u(x, t) &= \exp[i\xi(x - bt) - a|\xi|^2t] \\
\text{wave speed} &= b \text{ (property of the medium). amplitude damped with } \exp[-a|\xi|^2t].
\end{align*}
\]

Dispersion:

\[
\begin{align*}
\partial_t u + i\Delta_x u &= 0, \\
v &= v(\xi) = \xi
\end{align*}
\]

wave speed dependent on the frequency!

hyperbolicity \iff \( v \) only medium dependent

dispersivity \iff \( v \) frequency dependent

parabolicity \iff exponential damping

Systems: \( \partial_t u + \partial_{x_j}[A_{j}u] \)

Difference methods always exhibit all three features, regardless of the type of equation!
Difference methods for advection diffusion equations

A linear example:

\[
\partial_t u + \partial_x [-a \partial_x u + bu] = 0, \quad u(x, 0) = u^I(x)
\]

Discretization: Grid: \( x_j = jh, \quad j \in \mathbb{Z}, \quad t_n = nk, n \in \mathbb{N} \), approximate \( u(x_j, t_n) \approx U(x_j, t_n) \)

Difference approximation:

\[
U(x_j, t_{n+1}) = U(x_j, t_n) + \frac{k}{h^2}[(a + \frac{h}{2}b)U(x_{j-1}, t_n) - 2aU(x_j, t_n) + (a - \frac{h}{2}b)U(x_{j+1}, t_n)]
\]

\[
U(x_j, t_0) = u^I(x_j)
\]

advance in time: \( U(*, t_n) \rightarrow U(*, t_{n+1}), \quad t_{n+1} - t_n = k \)

Notation: The translation operator:

\[
Tu(x) = u(x + h), \quad T^{-1}u(x) = u(x - h)
\]
\[ U(x, t + k) = \{1 + \frac{k}{h^2}[(a + \frac{h}{2}b)T^{-1} - 2a + (a - \frac{h}{2}b)T]\}U(x, t) \]

Linear wave ansatz: \( U(x, t) = \exp[i\xi(x - vt)] \)

\[
\exp[-i\xi v k] = 1 - a\frac{4k}{h^2}\sin^2(\frac{\xi h}{2}) - ib\frac{k}{h}\sin(\xi h)
\]

\[ e^{-i\xi v} = qe^{-i\xi v_r}, \ v_r \in \mathbb{R} \]

\[ U(x, t) = q^t \exp[i\xi(x - v_r t)] \]

\[ q^{2k} = [1 - a\frac{4k}{h^2}\sin^2(\frac{\xi h}{2})]^2 + [b\frac{k}{h}\sin(\xi h)]^2, \ \tan(\xi k v_r) = \frac{b\frac{k}{h}\sin(\xi h)}{1 - a\frac{4k}{h^2}\sin^2(\frac{\xi h}{2})} \]

Wave speeds always dependent on frequencies.

For \( a = 0 \) waves are always amplified (instability!)

Difference discretizations are always dispersive.

Explicit difference discretizations always need some amount of diffusion to be stable (artificial diffusion).
\[ a = k\alpha, \quad c = \frac{k}{h}, \quad z = \sin\left(\frac{\xi h}{2}\right) \]

\[ q^{2k} = [1 - 4\alpha c^2 z^1]^2 + b^2 c^2 z^2 (1 - z^2) \leq 1, \quad \forall z \in [0, 1] \]

\[ \Rightarrow b^2 c^2 \leq 2akc^2 < 1 \]

CFL condition: \( \frac{k|b|}{h} < 1 \)

Artificial diffusion: \( a = O(k) \)

Also true for nonlinear problems and problems with non-constant coefficients!
Characteristics for linear and scalar problems:

\[ \partial_t u + a(x, t) \partial_x u = 0, \quad u(x, 0) = u^I(x) \]

Characteristic:
Try to find a solution of the form \( u(\xi(t), t) = \text{const} \)

\[ \frac{d}{dt} \xi_{ys}(t) = a(\xi_{ys}, t), \quad \xi_{ys}(s) = y, \]

\[ \Rightarrow \quad \frac{d}{dt} u(\xi_{ys}(t), t) = 0, \quad u(\xi_{ys}(t), t) = u(y, s) \]

**forward:** \( u(\xi_{y0}(t), t) = u^I(y) \),

**backward:** The solution \( u(x, t) \) is given by

\[ u(y, s) = u(\xi_{ys}(t), t) = u^I(\xi_{ys}(0)), \]


Conservative formulation:
\[ \partial_t u + \partial_x[a(x, t)u] = 0, \quad u(x, 0) = u^I(x) \]

Characteristic:
\[ \frac{d}{dt}\xi_{ys}(t) = a(\xi_{ys}, t), \quad \xi_{ys}(s) = y, \]
\[ \Rightarrow \quad \frac{d}{dt}u(\xi_{ys}(t), t) = -u(\xi_{ys}(t), t)\partial_xa(\xi_{ys}(t), t), \]
\[ \frac{d}{dt}\xi_{ys}(t) = a(\xi_{ys}, t), \quad \frac{d}{dt}v_{ys}(t) = -v_{ys}\partial_xa(\xi_{ys}, t), \]
\[ v_{ys}(t) = u(\xi_{ys}(t), t), \quad \xi_{ys}(s) = y, \quad v_{ys}(s) = u(y, s), \]
forward:
\[ \frac{d}{dt}\xi_{y0} = a(\xi_{y0}, t), \quad \frac{d}{dt}v_{y0} = -v_{y0}\partial_xa(\xi_{y0}, t), \]
\[ \xi_{y0}(0) = y, \quad v_{y0}(0) = u^I(y), \]

Characteristics can never intersect for linear problems!
Find the solution: Given \((y, s)\)
backward solve:
\[
\xi'_ys(t) = a(\xi ys, t), \quad \xi ys(s) = y \rightarrow \xi ys(0) := \xi^0 ys
\]
forward solve:
\[
v'_ys(t) = -\partial_x a(\xi ys, t)v ys, \quad v ys(0) = u_I(\xi^0 ys) \rightarrow v ys(s) = u(\xi ys(s), s) = u(y)
\]
Example: \(\partial_t u + \partial_x [xu] = 0\)

Nonlinear problems:
\[
\partial_t u + \partial_x f(x, t, u) = 0, \quad u(x, 0) = u^I(x)
\]
\[
\frac{d}{dt} \xi ys = \partial_u f(\xi ys, t, v ys), \quad \frac{d}{dt} v ys = -\partial_x f(\xi ys, t, v ys),
\]
For nonlinear problems characteristics can intersect. The solution will develop a discontinuity (shock)!

Example: Burger’s equation \(\partial_t u + \partial_x \frac{u^2}{2} = 0, \quad u(x, 0) = u^I(x)\)
THE CONCEPT OF WEAK SOLUTIONS

\[ \partial_t u + \partial_x f(u) = 0, \quad u(x,0) = u^I(x) \quad (1) \]

\( u \) discontinuous \( \Rightarrow \) need to re-define derivatives
\( \psi \) sufficiently smooth test function, \( \psi \in C_0^\infty \)

Definition: \( u \) is a weak solution of (1) iff

\[ \int_0^\infty dt \int dx \left[ u \partial_t \psi + f(u) \partial_x \psi \right] = -\int dx \left[ \psi(x,0) u^I(x) \right] \quad \forall \psi \]

Remark: \( u \) differentiable \( \Rightarrow \) \( u \) classical solution.

Example:

\[ \partial_t u + \partial_x [H(x)u + \frac{1}{2}H(-x)u] = 0, \quad u(x,0) = u^I(x) \]

THE RANKINE - HUGONIOT CONDITION
Relates shockheight to shockspeed if there is a shock.

\[ \partial_t u + \partial_x f(u) = 0, \]

Assume:
1. \( u \) differentiable away from the shock curve \( x = \gamma(t) \).
2. \( u \) is a weak solution.

Implies:

\[ \gamma'(t)(u_+ - u_-)(\gamma(t), t) = (f_+ - f_-)(\gamma(t), t) \]

Example: Traffic flow

\[ \partial_t u + \partial_x \left[ c \left( 1 - \frac{u}{u_0} \right) u \right] = 0, \quad u(x, 0) = u^I(x) = \begin{cases} u_0 & \text{for } x > 0 \\ \frac{1}{2}u_0 & \text{for } x < 0 \end{cases} \]
GODUNOV’S METHOD

Scalar problems:

\[ \partial_t u + \partial_x f(u) = 0 \]

Equation for averages:

\[ U_n^j = \frac{1}{h} \int_{x_j}^{x_{j+1}} u(x, t_n) dx, \quad F_n^j = \frac{1}{k} \int_{t_n}^{t_{n+1}} f(u(x_j, t)) dt. \]

\[ \Rightarrow U_{n+1}^j - U_n^j + \frac{k}{h} (F_{n+1}^j - F_n^j) = 0 \]

Up to here exact!

Numerical approximation: compute \( F_n^j \) from \( U_n^{j-1}, U_n^j \)

Piecewise constant approximation: Assume \( u(x, t_n) = U_n^j \) for \( x_j < x < x_{j+1} \).

Solve problem with piecewise constant initial data (the Riemann problem) exactly.
\[ \partial_t u + \partial_x f(u) = 0, \quad u(x,0) = \begin{cases} U_L & \text{for} \ x < 0 \\ U_R & \text{for} \ x > 0 \end{cases} \]

Compute \( F = \frac{1}{k} \int_0^k f(u(0,t))dt \).

Characteristics: \( x' = f'(u), \ u' = 0 \)

Shock curve according to Rankine - Hugoniot:
\[ \gamma'(U_R - U_L) = f(U_R) - f(U_L) \]

Case 1: \( f'(U_R) > 0, f'(U_L) > 0 \) \Rightarrow \( F = f(U_L) \)
Case 2: \( f'(U_R) < 0, f'(U_L) < 0 \) \Rightarrow \( F = f(U_R) \)
Case 3: \( f'(U_R) < 0 < f'(U_L) \) and \( \gamma' > 0 \) \Rightarrow \( F = f(U_L) \)
Case 4: \( f'(U_R) < 0 < f'(U_L) \) and \( \gamma' < 0 \) \Rightarrow \( F = f(U_R) \)
Case 5: \( f'(U_L) < 0 < f'(U_R) \): rarefaction wave and the Bärenblatt solution.

THE BÄRENBLATT SOLUTION
\[ \partial_t u + \partial_x f(u) = 0, \text{ set } u(x, t) = g\left(\frac{x}{t}\right) \Rightarrow f'(g(z)) = z \]

\[ u(x, t) = (f')^{-1}\left(\frac{x}{t}\right) \text{ gives continuous solution in Case 5!} \]

Case 5: \( f'(U_L) < 0 < f'(U_R) \Rightarrow F = f(u_s) \text{ with } f'(u_s) = 0.\)

\( u_s \) is called the sonic point and \( u(x, t) = (f')^{-1}\left(\frac{x}{t}\right) \) is called a rarefaction wave.
Godunov:
\[ \gamma' := \frac{f(U_j^n) - f(U_{j-1}^n)}{U_j^n - U_{j-1}^n} \]

Case 1: \( f'(U_j^n) > 0, f'(U_{j-1}^n) > 0 \) \( \Rightarrow \) \( F_j^n = f(U_{j-1}^n) \)
Case 2: \( f'(U_j^n) < 0, f'(U_{j-1}^n) < 0 \) \( \Rightarrow \) \( F_j^n = f(U_j^n) \)
Case 3: \( f'(U_j^n) < 0 < f'(U_{j-1}^n) \) and \( \gamma' > 0 \) \( \Rightarrow \) \( F_j^n = f(U_{j-1}^n) \)
Case 4: \( f'(U_j^n) < 0 < f'(U_{j-1}^n) \) and \( \gamma' < 0 \) \( \Rightarrow \) \( F_j^n = f(U_j^n) \)
Case 5: \( f'(U_{j-1}^n) < 0 < f'(U_j^n) \) \( \Rightarrow \) \( F_j^n = f(u_s) \) with \( f'(u_s) = 0. \)

SYSTEMS AND APPROXIMATE RIEMANN SOLVERS

\[ \partial_t u + \partial_x f = 0, \quad u \in \mathbb{R}^N \]
\[ U_j^{n+1} - U_j^n + \frac{k}{h}(F_{j+1}^n - F_j^n) = 0, \quad F_j^n = F(U_{j-1}^n, U_j^n, v_0), \]
Flux Approximation

Solution with approximate flux function $\tilde{f}_{UL,UR}$

$$\partial_t v + \partial_x \tilde{f}_{UL,UR}(v) = 0, \quad v_0 := v(x_j, t)$$

Requirements on $F$ and $\tilde{f}_{UL,UR}$:

Conservation:

$$F(U_L, U_R, U_L) = f(U_L), \quad F(U_L, U_R, U_R) = f(U_R),$$

Shock speeds:

$$\tilde{f}_{UL,UR}(U_R) - \tilde{f}_{UL,UR}(U_L) = f(U_R) - f(U_L) \quad (2)$$

Set:

$$F(U_L, U_R, v) = \tilde{f}_{UL,UR}(v) + f(U_L) - \tilde{f}_{UL,UR}(U_L) \quad (3)$$

$$= \tilde{f}_{UL,UR}(v) + f(U_R) - \tilde{f}_{UL,UR}(U_R)$$
Step 1: Find $\bar{f}_{UL,UR}$ satisfying (2)
Step 2: Solve approximate conservation law for $v_0$.
Step 3: Compute $F$ according to (3).

Approximate Riemann solvers:

$$\partial_t v + \partial_x \bar{f}_{UL,UR}(v) = 0, \quad v(x,0) = U_L, \ x < 0, \quad v(x,0) = U_R, \ x > 0$$

Linear approximation: $\bar{f}_{UL,UR}(v) = A(U_L, U_R)v$

$A$ : Roe matrix

Shock speeds:

$$A(U_L, U_R)(U_R - U_L) = f(U_R) - f(U_L)$$

diagonalize $A$ and solve $N$ scalar linear problems.

**SOLVING THE CONSTANT COEFFICIENT RIEMANN PROBLEM FOR SYSTEMS**
\[ A(U_L, U_R) \tilde{r}_m = \tilde{\lambda}_m \tilde{r}_m, \quad Df(U_L) r^L_m = \lambda^L_m r^L_m, \quad Df(U_R) r^R_m = \lambda^R_m r^R_m \]

expand \( U_L, U_R \) in eigenvectors

\[ U_L = \sum_m w^L_m \tilde{r}_m, \quad U_R = \sum_m w^R_m \tilde{r}_m, \]

solve

\[ \partial_t w_m + \tilde{\lambda}_m \partial_x w_m = 0, \quad w_m(x, 0) = \begin{cases} w^L_m & \text{for } x < 0 \\ w^R_m & \text{for } x > 0 \end{cases} \]

\[ w_m(0, t) = \begin{cases} w^L_m & \text{for } \tilde{\lambda}_m > 0 \\ w^R_m & \text{for } \tilde{\lambda}_m < 0 \end{cases}, \quad v(0, t) = \sum_m w_m(0, t) \tilde{r}_m \]
Problem:
Only shock solutions, no rarefaction waves.

except: $\lambda_m^L < 0 < \lambda_m^R$ (rarefaction wave)
replace by two shocks: $w_m(0, t) = w^s$

conservation:

$$w_m^L(k\lambda_m^L+C)+kw_m^s(\lambda_m^R-\lambda_m^L)+w_m^R(C-k\lambda_m^R) = w_m^L(k\tilde{\lambda}_m+C)+w_m^R(C-k\tilde{\lambda}_m)$$

$$w_m^s = \frac{w_m^L(\tilde{\lambda}_m-\lambda_m^L) + w_m^R(\lambda_m^R-\tilde{\lambda}_m)}{\lambda_m^R - \lambda_m^L}$$

This is sometimes called the 'sonic fix'.

SUMMARY

Roe matrix:

$$A_j^n = A(U_{j-1}^n, U_j^n), \quad A_j^n(U_j^n - U_{j-1}^{n-1}) = f(U_j^n) - f(U_{j-1}^n)$$
\[ F^n_j = f(U^n_{j-1}) + A^n_j(v^0 - U^n_{j-1}), \quad v^0 = \sum_m w^0_m \tilde{r}_m \]

\[ w^0_m = \begin{cases} w^L_m & \text{for } \tilde{\lambda}_m > 0 \\ w^R_m & \text{for } \tilde{\lambda}_m < 0 \end{cases}, \quad U^n_{j-1} = \sum_m w^L_m \tilde{r}_m, \quad U^n_j = \sum_m w^R_m \tilde{r}_m \]

except for the sonic fix:

\[ Df(U^n_j)r^n_{jm} = \lambda^n_{jm} r^n_{jm}, \quad m = 1, ..., N \]

If \( \lambda_{j-1,m} < 0 < \lambda^n_{j,m} \) then

\[ w^0_m = \frac{w^L_m(\tilde{\lambda}_m - \lambda^n_{j-1,m}) + w^R_m(\lambda^n_{j,m} - \tilde{\lambda}_m)}{\lambda^n_{j,m} - \lambda^n_{j-1,m}} \]

**ONE WAY TO FIND A ROE MATRIX**

\[ A(U_L, U_R) = \int_0^1 Df(U_L + s(U_R - U_L)) ds \]
Example 1: Burger’s equation, \( f(u) = \frac{1}{2} u^2 \)

Example 2: Isothermal flow

\[
f(\rho, \phi) = \left( \frac{\phi^2}{\rho} + \rho \right),
\]

\[
A(U_L, U_R) = \begin{pmatrix} 0 & 1 \\ 1 - v^2 & 2v \end{pmatrix}, \quad v = \frac{\sqrt{\phi_L^2 / \rho_L} + \sqrt{\phi_R^2 / \rho_R}}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \quad v_{LR} = \frac{\phi_{LR}}{\rho_{LR}}
\]
ODE Methods and Their PDE Counterparts

ODE: \( \frac{du}{dt} = f(u, t) \)

Euler’s method (explicit): \( U(t + k) = U(t) + kf(U, t) \)

Higher order methods I: Runge Kutta

Idea: Write in integral form

\[ u(t + k) = u(t) + k \int_0^1 f(u, t + sk) \, ds \]

Use integration rule for the integral

Euler: \( \int_0^1 f(u, t + sk) \, ds \approx f(U, t) \)

Second order: \( \int_0^1 f(u, t + sk) \, ds \approx f(U_{1/2}, t + k/2) \); compute
\[ u_{1/2} \text{ by intermediate Euler step} \]

\[ U_{1/2} = U(t) + \frac{k}{2} f(U, t), \quad U(t + k) = U(t) + kf(U_{1/2}, t + \frac{k}{2}) \]

Equivalent for hyperbolic PDE's: Lax-Wendroff in staggered grid formulation

\[ \partial_t u + \partial_x f(u) = 0, \quad U(t+k) = U(t) - \frac{k}{h} \left( T^{1/2} - T^{-1/2} \right) f(U(t+\frac{k}{2})) \]

\[ U(t + \frac{k}{2}) = \frac{1}{2} \left( T^{1/2} + T^{-1/2} \right) U(t) - \frac{k}{2h} \left( T^{1/2} - T^{-1/2} \right) f(U(t)) \]

Higher order methods have less artificial diffusion!

\[ \partial_t u + b \partial_x u = 0: \text{Godunov: } \frac{C |b|}{2} h^2 \partial_x u \quad \text{Lax - Wendroff: } \frac{C^2 b^2}{2} h^2 \partial_x u, \]

\[ C = \frac{k}{h} \]

Higher order methods II: Multi-step
\[
\frac{du}{dt} = f(u, t), \quad U(t + k) = \sum_{s=0}^{S} \alpha_s U(t - sk) + k\beta_s f(U, t - sk)
\]

Problem 1: needs start up!

Stability:

\[
U(nk) = g^n U(0), \quad g^{n+1} = \sum_{s=0}^{S} \alpha_s g^{n-s} + k...
\]

\(g\) solution of polynomial of degree \(S + 1\), \(S + 1\) roots!

Example: Leapfrog for \(\partial_t u + \partial_x f(u)\)

\[
U(t + k) = U(t - k) - \frac{bk}{h} (T - T^{-1}) f(U(t)),
\]

growth function \(g\) for \(\partial_t f + b\partial_x f = 0\):

\[
g^2 = 1 + 2i gbC \sin(\xi h), \quad C = \frac{k}{h}
\]

For \(|b|C \leq 1\): \(|g_{12}| = 1\), no artificial diffusion!!! (weakly unstable)
MONOTONICITY AND THE MAXIMUM PRINCIPLE

Difference scheme for general nonlinear problem

\[ U_j(t + k) = g_j(U_1(t), .., U_N(t)) \]

Consistency ⇒

\[ g_j(u, ...., u) = u, \ \forall j \ \text{(or} \ g_j(u, ...., u) = u(1 + O(k)), \ \forall j \ \text{in the presence of source terms)}. \]

Lemma:

\[ \frac{\partial g_j}{\partial u_k} \geq 0 \ \forall j, k \ \Rightarrow \ \|U(t + k)\|_\infty \leq \|U(t)\|_\infty, \]

Implicit schemes:

\[ g_j^+(U(t + k)) = g_j^-(U(t)) \]

\[ g_j^+(.., U_s(t), ..) = g_j^+(.., U_s(t), ..), \ \forall j \]
Define:

\[ g^+_j(v) = G^+_j(\ldots, v, v, \ldots), \quad g^-_j(v) = G^-_j(\ldots, v, v, \ldots) \]

Assume:

(A1) \[ \frac{\partial G^+_j}{\partial U_s} \leq 0 \text{ for } j \neq s, \quad \frac{\partial G^-_j}{\partial U_s} \geq 0 \quad \forall j, s \]

(A2) \[ g^+_j(v) \geq g^-_j(w) \Rightarrow v \geq w \quad \forall j \]

Lemma: Assumptions (A1),(A2) imply \[ \|U(t+k)\|_\infty \leq \|U(t)\|_\infty \]

Monotone schemes build (nonlinear) averages over the solution with nonnegative coefficients!

Example: Lax - Wendroff

Methods of order higher than order 1 cannot be monotone!

THE TVD PROPERTY
The discrete $W_1^\infty$ norm: $\int |\partial_x u| \, dx$

Non-oscillatory schemes:

**Definition:** A scheme is TVD $\iff$

$$\sum_j |(T - 1)U_j^{n+1}| \leq \sum_j |(T - 1)U_j^n|$$

The linear case:
Write scheme solely in terms of derivatives.

$$U^{n+1} = U^n - T^{-1}(A^n V^n) + B^n V^n, \quad V^n := (T - 1)U^n$$

**Theorem:**

$$A, B \geq 0, \quad A + B \leq 1 \quad \Rightarrow \quad TVD$$
LECTURE 5

FLUX LIMITER METHODS

\[ U^{n+1} = U^n - c(T - 1)F^n \]

Idea:
Lower order method with numerical flux \( F^L \). Higher order method with numerical flux \( F^H \). Lower order method non-oscillatory.

Combine:

\[ F = F^L + \phi(F^H - F^L) \]

Smooth part: \( \phi \approx 1 \); Non-oscillatory otherwise, i.e. total method is TVD.
Total method formally only first order but higher order in smooth regions.

**THE FLUX LIMITER**

Condition 1: (Smooth order)

\[ \Phi = \phi\left(\frac{(1 - T^{-1})U}{(T - 1)U}\right), \quad \phi(1) = 1 \]

Condition 2: Choose \( \phi \) such that total method is TVD.

Derive \( \phi \) for linear case and use in general.
\[ \partial_t u + a \partial_x u = 0, \quad a > 0 \]

Example: Upwind and Lax Wendroff

\[ F^L = aT^{-1}U, \quad F^H = \frac{a}{2}(1 + T^{-1})U - \frac{ca^2}{2}(1 - T^{-1})U \]

\[ F^H - F^L = \frac{a}{2}(1 - ac)(1 - T^{-1})U \]

\[ U^{n+1} = \{1 - \nu(1 - T^{-1})[1 + \frac{\phi}{2}(1 - \nu)(T - 1)]\}U^n, \quad \nu := ac \]

\[ U^{n+1} = U^n - \nu[1 - \frac{(1 - \nu)}{2}T^{-1}\phi]T^{-1}V^n - \nu\frac{\phi}{2}(1 - \nu)V^n, \quad V = (T-1)U \]

\[ A = \nu\{1 - \frac{(1 - \nu)}{2}\phi\} + \nu\frac{T\phi}{2}(1 - \nu)\frac{TV}{V} \]

\[ A = \nu\{1 + \frac{(1 - \nu)}{2}(-\phi + T\phi\frac{TV}{V})\} \]
$$A \geq 0 \implies \left| \frac{\phi(\theta_j)}{\theta_j} - \phi(\theta_{j-1}) \right| \leq 2, \quad \theta_j = \frac{T^{-1}V}{V}$$

Two standard choices:

1. Superbee:
   $$\phi(\theta) = \max\{0, \min\{1, 2\theta\}, \min\{\theta, 2\}\}$$

2. Van Leer:
   $$\phi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|}$$
LECTURE 6

BOUNDARY CONDITIONS AND GHOSTPOINTS FOR HYPERBOLIC SYSTEMS

\[
\partial_t u + \partial_x [A(x, t)u] = 0, \quad x \geq 0, \quad Bu(0, t) = u_b,
\]
\[u \in \mathbb{R}^N, \quad B \in \mathbb{R}^{K \times N}\]

Relation of \(B\) to \(A\): Influx given in terms of outflux!

Diagonalization:

\[
A(0, t) = EDE^{-1}, \quad D = \begin{pmatrix} D_+ & 0 \\ 0 & D_- \end{pmatrix}, \quad D_+ > 0, \quad D_- \leq 0
\]

Important!! \(\text{dim}(D) = K \times K\)

Projections: partition \(E\)

\[
E = (E_+, E_-), \quad E_+ : N \times K, \quad E^{-1} = \begin{pmatrix} R_+ \\ R_- \end{pmatrix}, \quad R_+ : K \times N
\]
Necessary: $\exists(BE_+)^{-1}$

Ghostpoint:

$$R_- U(-h, t) = R_-(2U(0, t) - U(h, t))$$

$$R_+ U(-h, t) = (BE_+)^{-1} [u_b - BE_- R_-(2U(0, t) - U(h, t))]$$

$$U(-h, t) = E \left( \begin{pmatrix} (BE_+)^{-1} \\ 0 \end{pmatrix} \right) u_b + E \left( \begin{pmatrix} -(BE_+)^{-1} BE_- R_- \\ R_- \end{pmatrix} \right) (2U(0, t) - U(h, t))$$