Robust Optimal Control of Demand-driven Supply Networks

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Overview

1. Supply Networks as Discrete Dynamical Systems
2. Constrained Robust Optimal Control
3. Computational Results
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Example: Beer Game

A model of a four-stage beer supply chain

- ... to study the formation of control rules of individual as potential sources of the bullwhip effect [Sterman 1989]
- ... showing different kinds of chaotic behavior for certain rules
Dynamics of Nodes

Dynamics of a node \( v \in V \) is given by the map

\[
x^{(v)}(t + 1) = x^{(v)}(t) + e^{\delta(v)} \cdot u_S^{(v)}(t) - e^{\hat{\delta}(v)} \cdot z_S^{(v)}(t),
\]

\[
y_R^{(v)}(t) = z_R^{(v)}(t),
\]

\[
y_S^{(v)}(t) = z_S^{(v)}(t),
\]

where \( e^k \in \mathbb{R}^k \) denotes a row vector with all components equal to one.
Additional Nodes (Delay Elements)

\[
x^{(w)}(t + 1) = u^{(w)}_S(t), \quad x^{(w')} (t + 1) = u^{(w')}_R(t),
\]
\[
y^{(w)}_S(t) = x^{(w)}(t), \quad y^{(w')}_R(t) = x^{(w')} (t).
\]
Additional Nodes (Sources and Sinks)

Sinks (end consumers):

\[ y_{R}^{(v_C)}(t) = d^{(v_C)}(t) \]  \hspace{1cm} (3)

Sources:

\[ y_{S}^{(v_M)}(t) = u_{R}^{(v_M)}(t) \]  \hspace{1cm} (4)

for all \( v_{M} \in V_{M} := \{v \in V : \delta^{(v_M)} = \delta'_{v_M} = 0\}\).

Source and sink nodes have no internal state variables.
Resulting Description

By connecting the inputs and outputs of adjacent nodes: First-order difference equation of the form

\[ x(t + 1) = Ax(t) + Bu(t) + Ed(t), \]  

(5)

with matrices \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times n_u} \), and \( E \in \mathbb{R}^{n \times n_d} \), where

\[ u(t) := [z_1(t) \cdots z_{|\mathbb{V}_z|}(t)]^T \]  

(6)

are the remaining free control variables.
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Robust Optimal Control Model

Assumption: Stochastic input $d(t) \in \mathcal{D} \subset \mathbb{R}^{nd}$ is bounded.

Determine recursively the optimal cost function value $J^*$ and corresponding optimal control input $u$ as explicit functions of the state $x$ (closed-loop feedback control):

$$J^*(k)(x^{(k)}) = \min_{u^{(k)}} J(k)(x^{(k)}, u^{(k)})$$

s.t. \[
\begin{align*}
Fx^{(k)} + Gu^{(k)} &\leq g \\
Ax^{(k)} + Bu^{(k)} + Ed^{(k)} &\in \chi^{(k)}
\end{align*}
\] \forall d^{(k)} \in \mathcal{D},

$$J(k)(x^{(k)}, u^{(k)}) = \max_{d^{(k)} \in \mathcal{D}} ||Qx^{(k)}||_1 + ||Ru^{(k)}||_1 + J^*(k+1)(Ax^{(k)} + Bu^{(k)} + Ed^{(k)})$$

for $k = K - 1, \ldots, 0$, where the set of feasible states is

$$\chi^{(k)} = \{ x \in \mathbb{R}^n : \exists u \in \mathbb{R}^{nu} \text{ with } Fx + Gu \leq g \text{ and } Ax + Bu + Ev \in \chi^{(k+1)} \forall d \in \mathcal{D} \}.$$
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Example 1: Beergame, One Stage

\[ x(t + 1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} d(t). \]

Feasible region:

\[ x_1 + x_2 \geq 8, \]
\[ x_1 + x_2 + x_3 \geq 16, \]
\[ x_1 + x_2 + x_3 + x_4 \geq 24 \]  \hspace{1cm} (8)

Optimal Control:

\[ z^{(B)}_R(x) = u^*(x) = \max\{32 - x_1 - x_2 - x_3 - x_4, 0\}. \]  \hspace{1cm} (9)
Example 2: Beergame, All 4 Stages

Feasible region:

\[ 32 \leq x^{(B)} + x^{(W)} + \sum_{i=1}^{2} (x^{(\sigma i_2)} + x^{(\sigma i_1)}), \]
\[ 40 \leq x^{(B)} + x^{(W)} + \sum_{i=1}^{2} (x^{(\sigma i_2)} + x^{(\sigma i_1)}) + x^{(\rho_{21})}, \]
\[ 48 \leq x^{(B)} + x^{(W)} + x^{(D)} + \sum_{i=1}^{3} (x^{(\sigma i_2)} + x^{(\sigma i_1)}), \]
\[ 56 \leq x^{(B)} + x^{(W)} + x^{(D)} + \sum_{i=1}^{3} (x^{(\sigma i_2)} + x^{(\sigma i_1)}) + x^{(\rho_{31})}, \]
\[ 64 \leq x^{(B)} + x^{(W)} + x^{(D)} + x^{(F)} + \sum_{i=1}^{4} (x^{(\sigma i_2)} + x^{(\sigma i_1)}), \]

and a resulting optimal control law is (9) for \( u_1^* \) and

\[ u_2^*(x) = \max\{48 - x^{(B)} - x^{(W)} - \sum_{i=1}^{2} (x^{(\sigma i_2)} + x^{(\sigma i_1)}) - x^{(\rho_{21})}, 0\}, \]
\[ u_3^*(x) = \max\{64 - x^{(B)} - x^{(W)} - x^{(D)} - \sum_{i=1}^{3} (x^{(\sigma i_2)} + x^{(\sigma i_1)}) - x^{(\rho_{31})}, 0\}, \]
\[ u_4^*(x) = \max\{72 - x^{(B)} - x^{(W)} - x^{(D)} - x^{(F)} - \sum_{i=1}^{4} (x^{(\sigma i_2)} + x^{(\sigma i_1)}), 0\}, \]
Example 3: One Retailer, Two Suppliers

\[ x(t + 1) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} d(t). \]

Feasible region is given by \( x_1 + x_2 + x_4 \geq 8 \) and \( x_1 + x_2 + x_3 + x_4 \geq 10 \). Optimal control is

\[ u_1^*(x) = \min\{\max\{20 - x_1 - x_2 - x_3 - x_4, 0\}, 4\}, \]
\[ u_2^*(x) = \max\{16 - x_1 - x_2 - x_3 - x_4, 0\}. \]

This is a so-called dual base-stock policy.