Lagrange Multiplier Theorem For Optimal Control Problems

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Contents

Contents

The 'first' optimal control problem

An airplane should be moved in shortest time from y_1 to y_2. Initially and finally it should stand still. The aircraft can be accelerated or decelerate by a force.

Mathematical formulation for the position $y(t)$ of the airplane

$$\begin{aligned} \min T \text{ subject to} \qquad & \text{Cost functional} \\ m\, y''(t) = u(t), \qquad & \text{State equation} \\ y(0) = y_1, y(T) = y_2, y'(0) = y'(T) = 0, \qquad & \text{State equation} \end{aligned}$$

'Real–world' problems

A body should be heated up to certain temperature. Heating can be applied at every location.

Mathematical formulation for the temperature $y(x,t)$

$$\min \int_0^T \int_\Omega \frac{\alpha_1}{2} \big(y(t,x) - \bar{y}(t,x)\big)^2 + \frac{\alpha_2}{2}\, u^2(t,x)\,dxdt \qquad \text{Cost functional}$$

$$y_t - \Delta_x y = u \ \text{ in } \Omega \times (0,T) \qquad \text{State equation}$$

$$y(x,t) = 0 \ \text{ auf } \Gamma \times (0,T) \qquad \text{State equation}$$

$$y(x,0) = 0 \ \text{ in } \Omega \qquad \text{State equation}$$

Contents

Questions and Answers

$$\boxed{\min f(u, y) \text{ subject to } h(u, y) = 0}$$

- Existence of an optimal control u?
- Equation for computing u?

Reduction of the problem

$f(u, y) : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ is the cost functional $\quad (\int y^2 + u^2 dxdt)$

$h(u, y) = Bu - Ay$ is the state equation $\quad (\ y_t - \Delta y - u = 0)$

$$\boxed{\min_{u,y} f(u, y) \text{ sbj to } h(u, y) = Bu - Ay = 0}$$

Additional assumption: A regular $\quad$ (Solvability of pde)

$\rightarrow y = A^{-1}Bu \quad (y = (\partial_t - \Delta)^{-1}u)$

Yields reduced cost problem with reduced cost functional $\tilde{f}$

$$\min_u \tilde{f}(u) \text{ where } \tilde{f}(u) = f(u, A^{-1}Bu)$$

Existence

$$\boxed{\min \tilde{f}(u) \text{ where } \tilde{f}(u) = f(u, A^{-1}Bu)}$$

Weierstrass. $\tilde{f}$ continuous, $u \in U \subset \mathbb{R}^m \neq \emptyset$, U bounded and closed, then there exists one optimal control u.

Modification: U unbounded, then consider the cost functional by $\tilde{f}(u) + \lambda \|u\|_2^2$ with $\lambda > 0$. (Tychonoff regularization).

Equation for u (KKT System)

$$\boxed{\min f(u,y) \text{ subject to } h(u,y) = Bu - Ay}$$

Karush–Kuhn–Tucker. f, h continuously differentiable, u^*, y^* the local minimizer under the constraint $h(u,y) = 0$. h satisfies a constraint qualification (e.g. Dh has maximal row rank), then there exists multiplier λ^*, such that

$$\begin{aligned} Bu^* &= Ay^* && \text{State equation} \\ \nabla_y f(u^*, y^*) + A^T\lambda^* &= 0 && \text{Adjoint equation} \\ \nabla_u f(u^*, y^*) - B^T\lambda^* &= 0 && \text{Optimality condition} \end{aligned}$$

Example

- Optimal control of a diffusion equation in $\Omega \subset \mathbb{R}^n$ with $y = 0$ on $\partial\Omega$ and distributed control

$$-\Delta_x y = u$$

- PDE is well–defined for $y \in X := H_0^1(\Omega), dimX = \infty$ and reads

$$\int_\Omega \nabla_x y \cdot \nabla_x \phi \, dx = \int_\Omega u \, \phi \, dx \qquad \forall \phi \in H_0^1(\Omega)$$

- Define $\boxed{< a(y), \phi >:= \int_\Omega \nabla_x y \cdot \nabla_x \phi \, dx} \in H_0^{-1}(\Omega)$ for fixed $y \in H_0^1(\Omega)$
- Constraint is an operator equation

$$a(y) = b(u) \text{ in } \left(H_0^1(\Omega)\right)'$$

Example (cont'd)

$$\min_{y \in H^1, u \in L^2} \int_\Omega (y - y_d)^2 + u^2 dx \text{ sbj to } a(y) = b(u) \in H^{-1}$$

Translate concepts of finite–dimensional case

- Continuous cost function → Weak semi lowercontinuity
- Multiplier λ and scalar product → Element of dual and duality product
- DH has full row rank → Surjective operator $DH()$

Existence for infinite dimensional, non–compact sets U

Let $u \in U \subset X \neq \emptyset$ and U bounded, closed and convex. Let $S : X \to Y$ be a bounded, linear operator and $f(u, Su)$ weak lower semi–continuous and bounded from below. Then, there exists at least one optimal control u.

U bounded can be skipped, if the control is regularized.

Example is bounded from below and weak lower–semi continuous

$$f(u, y) = \|y - \bar{y}\|^2 + \lambda\|u\|^2 = \int_\Omega (y - \bar{y})^2 dx + \lambda \int_\Omega u^2 dx,$$

Lagrange multiplier theorem in infinite–dimensional spaces

Let X, Y be Hilbert spaces, $f : X \to \mathbb{R}$ and $H : X \to Y$ continuously Frechét differentiable. x^* is the local minimum of f with constraint $H(x) = 0 \in Y$ and $DH(x^*) \in L(X, Y)$ is surjective.
Then, there exists a $\lambda' \in Y'$ such that

$$f'(x^*) - < \lambda^*, H'(x^*) >= 0 \quad \textit{and} \quad H(x^*) = 0.$$

- Equation in X', i.e., $\forall x \in X : < f'(x^*), x > - < \lambda^*, H'(x^*)x >= 0.$
- Finite–dimensional: Each component of $Df + \lambda^T DH$ has to vanish

Reference: Luenburger, 1973

Contents

Sample problem

Control $u(t)$, initial data $\bar{\rho}_1$ and desired state $\bar{\rho}_2$ for $t \in (0, T)$ and $x \in (0, 1)$.

$$\min_{T,u} \int_0^T dt + \int_0^1 \frac{1}{2} \left(\rho(x,T) - \bar{\rho}_2\right)^2 dx, \text{ subject to}$$

$$\partial_t \rho + \partial_x \rho = 0,$$

$$\rho(x, 0) = \bar{\rho}_1,$$

$$\rho(0, t) = u(t)$$

Lagrangian for this problem

Lagrangian:

$$L(\rho, u, T, \lambda) := \int_0^T dt + \int_0^1 \frac{1}{2}(\rho(x,T) - \bar{\rho}_2)^2 \, dx + \int_0^1 \int_0^T (-\partial_t \lambda - \partial_x \lambda)\rho dx dt$$

Stationary points:

$$\partial_u L = 0 : \Longrightarrow \int_0^T \lambda(0,t)\delta u(t) dt = 0 \implies \lambda(0,t) = 0, t \in (0,T)$$

$$\partial_\rho L = 0 : \Longrightarrow$$

$$-\partial_t \lambda - \partial_x \lambda = 0, \; \lambda(1,t) = 0, \; \lambda(x,T) + (\rho(x,T) - \bar{\rho}_2) = 0$$

$$\partial_\lambda L = 0 : \Longrightarrow$$

$$\partial_t \rho + \partial_x \rho = 0, \; \rho(0,t) = u(t), \rho(x,0) = \bar{\rho}_1.$$

Solution to fixed T

Case 1. $T > 1$. Then, $\lambda(0,t) = 0$ for $t \in (0,T)$ and $\lambda(x,T) = \rho(x,T) - \bar{\rho}_2$ for $x \in (0,1)$ implies that in fact $\rho(x,T) = \bar{\rho}_2$ for $x \in (0,1)$. Then, $\lambda(x,t) = 0$ for all $(x,t) \in (0,1) \times (0,T)$. Furthermore, $\rho(x,t) = \bar{\rho}_2$ for $x \in (0,1)$ implies $u(t) = \bar{\rho}_2$ for $T \geq t \geq T-1$. In particular, there is no condition on $u(t)$ for $t < T-1$.

Case 2. $T \leq 1$. Then, $\lambda(0,t) = 0$ for $t \in (0,T)$ and $\lambda(x,T) = \rho(x,T) - \bar{\rho}_2$ for $x \in (0,1)$ implies that $\rho(x,T) = \bar{\rho}_2$ for $x \in (0,T)$. Them, $u(t) = \bar{\rho}_2$ for $t \in (0,T)$. In particular, $\lambda(x,t) = 0$ for $x \geq t$ and $\lambda(x,t) = 0$ for $x \leq t + (T-1)$. There is no condition on λ in the remaining strip in the $(x,t)-$plane of width $T-1$.

Reduced cost functional $G(T)$

- Definition

$$G(T) = \int_0^T dt + \int_0^1 \frac{1}{2} \left(\rho(x, T; T) - \bar{\rho}_2\right)^2 dx$$

where $(u(x, t; T), \rho(x, t; T), \lambda(x, t; T))$

- Values

$$G(T) = \begin{pmatrix} T & T \geq 1 \\ T + (1 - T)(\bar{\rho}_2 - \bar{\rho}_1) & T \leq 1 \end{pmatrix}$$

- Formal derivative

$$G'(T) = H(T - 1) + (\bar{\rho}_2 - \bar{\rho}_1) H(1 - T)$$

- Interpretation for the stationary point of $\partial_T L$?

Open questions

- Model of Armbruster and Ringhofer

$$\partial_t \rho + \partial_x v(\rho)\rho = 0$$

is the model equation for $v(\rho) = 1 - \int \rho dx$

- Differentiation of $v(\rho)$ with respect to ρ?
- Nonlinear dynamics or just $\partial_t \rho + \partial_x v(t)\rho = 0$

Thank you for your attention.