STOCHASTIC DYNAMICS OF LONG SUPPLY CHAINS WITH RANDOM BREAKDOWNS

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Abstract. We analyze the stochastic large time behavior of long supply chains via a traffic flow type random particle model. So items travel on a virtual road from one production stage to the next. Random breakdowns of the processors at each stage are modeled via a Markov process. The result is a conservation law for the expectation of the part density which holds on time scales which are large compared to the mean up and down times of the processors.

Key words. supply chains, traffic flow models, mean field theories, Boltzmann equation, fluid limits

AMS subject classifications. 65N35, 65N05

1. INTRODUCTION. Traffic flow type models for supply chains model the flow of items through the chain as conservation laws for an item density $\rho$, depending on time and a stage variable $x$. So, stage $x = 0$ denotes the raw material, and stage $x = 1$ denotes the finished product and the interval $[0, 1]$ models the intermediate stages of the production process, and plays the role of the ‘road’ in traffic flow theory. Traffic models have been used to model supply chains in [1, 2, 13, 5, 8] and, more recently to optimize them in [7, 9].

In previous work [3] we have developed a traffic flow type model for a chain of suppliers with a given capacity and throughput time. It is of the form

\begin{equation}
\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial F(x, t)}{\partial x} = 0, \quad F(x, t) = \min\{\mu(x), V(x)\rho\}
\end{equation}

Here $x$ denotes a continuous supplier index, i.e. the stage of the process. $\rho(x, t)$ denotes the density of parts in the supply chain. So, to compute the number of parts - the Work in Progress (or WIP) $W_{ab}(t)$ in a certain subset of processors, corresponding to an interval $(a, b)$ at a given time $t$, we have to compute $W_{ab}(t) = \int_{a}^{b} \rho(x, t) \, dx$. As long as the processors run below capacity, the movement of parts is given by the velocity $V$. So $\frac{dx}{V(x)}$ is proportional to the throughput time of the processor occupying the infinitesimal interval $dx$. The processors are assumed to have a finite capacity, meaning that they cannot process more that $\mu(x)dt$ parts in any infinitesimal time interval $dt$. So the variables in (1.1) have units of parts / stage for $\rho$, parts /time for $\mu$, and stage / time for $V$. We prescribe an, in general time dependent, influx of the form

\begin{equation}
F(0, t) = \lambda(t)
\end{equation}

for the conservation law (1.1).

Equation (1.1) is derived rigourously in [3] from a discrete recursion for the times each part arrives at each processor, and a limiting process for the number of parts.
and the number of processors \( M \to \infty \). However, this recursion relation is completely deterministic, and the supply chain is therefore assumed to work like an automaton. The goal of this paper is to include random behavior of the processors, i.e., random breakdowns and random repair times, into the model. We model the breakdown of processors by setting the capacity \( \mu(x,t) \) to zero. Thus, the model we consider consists of the equation (1.1), where \( \mu(x,t) \) is a time dependent random variable. To be more precise we assume \( \mu(x,t) \) to be piecewise constant in space and of the form

\[
\mu(x,t) = \sum_{m=0}^{M-1} \mu_m(t) \chi_{[\gamma_m, \gamma_{m+1}]}(x)
\]

where \( 0 = \gamma_0 < \ldots < \gamma_M = 1 \) denotes a partition of the stage interval \([0,1]\), corresponding to \( M \) processors, and the functions \( \mu_m(t) \), \( m = 0, \ldots, M-1 \) take on values of either \( \mu_m(t) = 0 \) or \( \mu_m = c_m \), where \( c_m \) denotes the capacity of the processor, in case it is running. For simplicity, we assume that the on / off switches are exponentially distributed in time. That is, we assume mean up and down times \( \tau_{\text{up}}^m \) and \( \tau_{\text{down}}^m \), and generate the random signal \( \mu_m(t) \) by the following algorithm:

- Assuming that at time \( t \) processor \( m \) has just switched from the off state to the on state, choose \( \Delta t_{\text{up}}^m \) and \( \Delta t_{\text{down}}^m \) randomly from the distributions \( dP[\Delta t_{\text{up}}^m = s] = \frac{1}{\tau_{\text{up}}^m} \exp(-\frac{s}{\tau_{\text{up}}^m})ds \) and \( dP[\Delta t_{\text{down}}^m = s] = \frac{1}{\tau_{\text{down}}^m} \exp(-\frac{s}{\tau_{\text{down}}^m})ds \).
- Set \( \mu_m(s) = c_m \) for \( t < s < t + \Delta t_{\text{up}}^m \) and \( \mu_m(s) = 0 \) for \( t + \Delta t_{\text{up}}^m < s < t + \Delta t_{\text{up}}^m + \Delta t_{\text{down}}^m \).
- At \( t = t + \Delta t_{\text{up}}^m + \Delta t_{\text{down}}^m \), the processor is turned on again and we repeat the above process.

This way we generate \( M \) random time dependent signals which produce the random capacity \( \mu(x,t) \) according to (1.3). For each realization of this process we solve one realization of the conservation law (1.1), modeling so the random breakdown of elements in the chain. To illustrate this, Figure 1.1 shows one realization of one of the signals \( \mu_m(t) \), switching between \( \mu_m = c_m = 2 \) and \( \mu_m = 0 \), and one realization of the solution of the corresponding conservation law. Note that the conservation law (1.1) exhibits, despite of its simple form, a rather interesting feature. Since the flux function \( F \) is uniformly bounded from above by \( \mu(x,t) \), it will necessarily become

![Figure 1.1](image-url)
discontinuous if the flux coming from the left exceeds this value. This can be the case if \( \mu(x, t) \) is discontinuous in the stage variable \( x \), which will certainly happen if \( \mu(x, t) \) is generated randomly by the algorithm above. Since mass has to be conserved, the discontinuity in the fluxes has to be compensated by \( \delta^- \) functions in the density \( \rho \). The temporary buildup of these \( \delta^- \) functions is what is observed in Figure 1.1.

The aim of this paper is to derive an evolution equation for the expectation \( \langle \rho(x, t) \rangle \) of the density \( \rho \) given by the stochastic process above. This provides us with a rather inexpensive way to estimate the behavior of long supply chains, with random breakdowns of individual processors, by solving directly one rather simple conservation law for the expectation. The main result of the present paper is, that the expectation \( \langle \rho(x, t) \rangle \) satisfies an initial boundary value problem for a conservation law of the form

\[
\begin{align*}
(1.4) & \\
& (a) \quad \partial_t \langle \rho(x, t) \rangle + \partial_x F_E(\bar{\tau}, C, V, \langle \rho \rangle) = 0, \quad F_E(\bar{\tau}, C, V, \langle \rho \rangle) = \bar{\tau}C[1 - \exp\left(-\frac{V(\rho)}{C}\right)] , \\
& (b) \quad F_E|_{x=0} = \lambda(t), \quad \langle \rho(x, 0) \rangle = 0
\end{align*}
\]

where the piecewise constant functions \( \bar{\tau} \) and \( C \) are given by

\[
\begin{align*}
(1.5) & \\
& \bar{\tau}(x) = \sum_{m=0}^{M-1} \chi[\gamma_m, \gamma_{m+1})(x) \frac{\tau_{up}^m}{\tau_{up}^m + \tau_{down}^m}, \quad C(x) = \sum_{m=0}^{M-1} \chi[\gamma_m, \gamma_{m+1})(x)c_m .
\end{align*}
\]

The result is derived in a limiting regime for large time scales and many parts and processors. So, it holds when the behavior of the chain, given by the stochastic version of (1.1), is considered on a time scale where a large number of parts arrive and the on / off switches of the processors happen very frequently. Similar models have been used on a heuristical basis, in the context of clearing functions, in \cite{11, 14}. Our result basically states two facts.

- For a large number of parts the function \( \min\{\mu, V\rho\} \) is, under the expectation, replaced by the function \( \mu[1 - \exp(-\frac{V(\rho)}{\mu})] \) which has the same limiting behavior for large and small densities (the limits \( \rho \to 0 \) and \( \rho \to \infty \)).
- The effect of the random on / off switches can be incorporated into the model by replacing \( \mu \) by the on- capacity \( c \) and multiplying the whole flux function by the average time \( \frac{\tau_{up}^{\text{avg}}}{\tau_{up}^{\text{avg}} + \tau_{down}^{\text{avg}}} \) the processor is on.

This paper is organized as follows: We prove the validity of the limiting equation (1.4) in a somewhat roundabout way. We first discretize one realization of (1.1) by a particle method, using the Lagrangian formulation of (1.1). Then we proceed to taking the appropriate limits. So, Section 2 is devoted to the the formulation of the particle method. The limiting behavior is derived in Section 3. We verify our results numerically, and demonstrate the basic premise of the method, namely that we can model the large time behavior of long chains accurately with the mean field equation (1.4) in Section 4. The proofs of Section 3 are given in the Appendix in Section 5.

2. PARTICLE FORMULATION. As mentioned in the introduction, we will derive the main result of this paper, the conservation law (1.4) for the expectation \( \langle \rho \rangle \), from a particle discretization of (1.1) in Lagrangian coordinates. Since we are going to employ a mean field theory approach to the particle model in the next
section, it is essential that the particle formulation of (1.1) is invariant under permutations of the particles. This will require some special considerations, and therefore we derive first a particle formulation of the deterministic problem, i.e. for one fixed realization of the random capacities \( \mu(x, t) \). In Section 2.2 we will then generalize this formulation to the random case.

2.1. The deterministic case. First, we reformulate problem (1.1) in Lagrangian coordinates. The transformation from Eulerian to Lagrangian coordinates is given in the usual manner by

\[
\begin{align*}
(a) \quad & \rho(x, t) = \int \delta(x - \xi(y, t)) \, dy, \\
(b) \quad & \rho(\xi(y, t), t) = -\frac{1}{\partial_y \xi(y, t)}.
\end{align*}
\]

where \( \xi(y, t) \) denotes the position of a particle with continuous index \( y \) at a time \( t \). The derivative \( \partial_y \xi = -\frac{1}{\rho} \) denotes the specific volume of the flow, i.e. the infinitesimal distance between two neighboring particles, and the minus sign indicates that we number the particles, at least initially, in order of their arrival, i.e. \( y_1 < y_2 \Rightarrow \xi(y_1) > \xi(y_2) \Rightarrow \partial_y \xi < 0 \) holds. Note, that (2.1)(b) only holds in the absence of caustics, that is as long as the particles stay ordered and do not overtake, whereas (2.1)(a) also holds in the presence of caustics. Using the transformation (2.1) the conservation law (1.1) becomes

\[
\partial_t \xi(y, t) = v(\xi, t) = \min \left\{ \frac{\mu(\xi, t)}{\rho(\xi, t)}, V(\xi) \right\}
\]

and reduces to a parameterized ordinary differential equation for the trajectories \( \xi(y, t) \). We consider a particle discretization of one realization of the stochastic version of the conservation law (1.1) by replacing \( \rho(x, t) \) by the measure corresponding to \( N \) particles

\[
\rho(x, t) \approx \Delta y \sum_{n=1}^{N} \delta(x - \xi_n(t))
\]

(2.3)

(where we choose the symbol \( \Delta y \) for the particle weight so as to be notationally consistent with (2.1)) and solve the system of ordinary differential equations

\[
\partial_t \xi_n(t) = v_n, \quad v_n \approx \min \left\{ \frac{\mu(\xi_n)}{\rho(\xi_n)}, V(\xi_n) \right\}
\]

(2.4)

There are still three aspects missing in the consistent formulation of the particle method (2.4).

- We have to decide on an appropriate weight \( \Delta y \) of each particle.
- We have to define initial conditions for the trajectories \( \xi_n \) to reproduce the boundary condition (1.2) of the conservation law.
- We still have do define how to compute the density \( \rho(x, t) \) at \( x = \xi_n \) from the particle ensemble.

To address the first issue, we assume that we start from an empty system. In this case, the total mass over all time is given by the integral over the influx \( \Lambda = \int_0^\infty \lambda(t) \, dt \), which we assume to be finite. To match this total mass \( \Lambda \) to the total mass in (2.3), we set \( \Delta y = \frac{\Lambda}{N} \).

To address the second issue, we note that the flux \( F \) of the Lagrangian formulation (2.2) is given by

\[
F(x, t) = \int \delta(x - \xi(y, t)) v(\xi) \, dy \quad \Rightarrow \quad \lambda(t) = F(0, t) = \int \delta(\xi(y, t)) v(\xi) \, dy.
\]
defining the initial condition $\xi(y, a(y)) = 0$ for the particles in the Lagrangian formulation (2.2) implies

\[
\lambda(t) = \int \delta(t - a(y)) \, dy \Rightarrow a^{-1}(t) = \int_0^t \lambda(s) \, ds.
\]

Thus, the arrival times $a(y)$ have to be chosen as the functional inverse of the monotone antiderivative of the influx function $\lambda(t)$, i.e. $a(y) = t \iff y = \int_0^t \lambda(s) \, ds$ has to hold. In the deterministic case, treated in [3], this implies that the arrivals $a(y)$ satisfy the ordinary differential equation

\[
\frac{da(y)}{dy} = \frac{1}{\lambda(a(y))}, \quad a(0) = 0.
\]

2.2. The random case. We now consider the stochastic process for the computation of the capacity variables $\mu_m(t)$ in (1.3). The assumption of an exponential distribution of the up and down times $\tau_{m}^{up}$ and $\tau_{m}^{down}$ implies a Markov process. This means, that at each infinitesimal time we can decide whether to switch the processor from on to off and back with a constant frequency $\omega_m(\mu) = \frac{1}{\tau_{m}^{up/down}}$. Thus the evolution of $\mu_m(t)$ can be expressed by the process

\[
\begin{align*}
(a) & \quad \mu_m(t + \Delta t) = (1 - r_m)\mu_m(t) + r_m(c_m - \mu_m(t)), & r_m = 0 \text{ or } r_m = 1, \\
(b) & \quad P[r_m = 1] = \Delta t\omega_m(\mu_m(t)), & P[r_m = 0] = 1 - \Delta t\omega_m(\mu_m(t)),
\end{align*}
\]

This means that at each infinitesimal time step $\Delta t$ we flip a coin and decide whether to switch, based on the probability $\Delta t\omega_m$, with the frequency $\omega_m$ given by

\[
\omega_m(0) = \frac{1}{\tau_{m}^{down}}, \quad \omega_m(c_m) = \frac{1}{\tau_{m}^{up}}.
\]

It is a standard exercise in the analysis of Monte Carlo methods (see c.f. [10]) that this algorithm results in exponentially distributed up and down times with means $\tau_{m}^{up}$ and $\tau_{m}^{down}$. We combine this with the motion of the particles $\xi_n$ given by

\[
\xi_n(t + \Delta t) = \xi_n(t) + \Delta t v_n(\xi(t), \vec{\mu}(t)),
\]

where the velocities $v_n$ depend on the whole particle ensemble $\vec{\xi} = (\xi_1, \ldots, \xi_N)$ and, in addition, on the random capacity vector $\vec{\mu}(t) = (\mu_1, \ldots, \mu_M)$. We still have to define a way to compute the density $\rho(\xi_n, t)$ from the particle ensemble $\vec{\xi}$. As stated before, the density $\rho$ is given, in terms of the particle formulation as the inverse of the specific volume, i.e. the distance of two neighboring particles. Formulating this in a way that is invariant under permutation of the particle index, we set

\[
\frac{1}{\rho(\xi_n, t)} = \min \{ \frac{\xi_k - \xi_n}{\Delta y} : \xi_k > \xi_n \}
\]

Note that, if the particles stay ordered in descending order, this reduces to $\frac{1}{\rho} = \frac{\xi_{n-1} - \xi_n}{\Delta y}$, which would be just the difference approximation to (2.1). The significance
of the formula (2.8) lies in the fact that it is also valid if particles pass each other and
the descending order is destroyed. Therefore, we choose the velocities \( v_n(\xi, \mu) \) as
\( (2.9) \)
\[
v_n(\xi(t), \mu(t)) = \min \left \{ V(\xi_n), \frac{\mu(\xi_n, t)}{\Delta y}(\xi_k - \xi_n) : \xi_k > \xi_n \right \}, \quad \mu(\xi_n, t) = \sum_m \chi_m(\xi_n) \mu_m(t)
\]
Turning to the boundary condition, we replace the influx density \( \lambda(t) \) in (2.5) by a
measure of the form
\( (2.10) \)
\[
\lambda(t) \approx \Delta y \sum_{n=1}^{N} \delta(t - a_n) = \frac{\Lambda}{N} \sum_{n=1}^{N} \delta(t - a_n)
\]
The goal here is again to formulate (2.10) in such a way, that the resulting particle
method is invariant under permutations of the particle index \( n \). We do so by random-
izing (2.10), and choose identically distributed random arrival times for each particle,
according to the probability distribution \( \lambda \Lambda \). So, we have
\( (2.11) \)
\[
\xi_n(a_n) = 0, \quad dP[a_n = t] = \frac{\lambda(t)}{\Lambda} dt, \quad \Lambda = \int_{0}^{\infty} \lambda(t) dt.
\]
(2.6)-(2.7), together with the definitions (2.9) and the initial condition (2.11) gives
now a complete set of rules to advance the particle positions \( \xi \) and the capacities \( \mu \)
from one time step to the next, and these rules are independent under permutations
of the particle index. We will reformulate the system once more, to essentially replace
the boundary condition (1.2) by an initial condition. This is really a technicality, and
the reason for it is that, in the next section, we will derive equations for the probability
density of the particle ensemble. To this end it is notationally more convenient to
deal with a fixed number of particles in the system, instead of particles which enter
at random times \( a_n \). So, instead of imposing the condition \( \xi_n(a_n) = 0 \), we move
the particles with an arbitrary, constant and deterministic, velocity - say \( V(0) \) - as
long as \( \xi_n(t) < 0 \) holds, and start them out at \( \xi_n(0) = -V(0)a_n \). So obviously
\( \xi_n(t) = V(0)(t - a_n) \) will hold for \( \xi_n < 0 \) and the particle will arrive at \( \xi_n = 0 \) at the
correct time.

So, in summary, the stochastic particle system, which will be analyzed in the next
section is of the following form.
Start out at \( t = 0 \) with
\( (2.12) \)
\[
(a) \quad \mu_m(0) = c_m, \quad m = 0, \ldots, M - 1,
(b) \quad \xi_n(0) = -V(0)a_n, \quad n = 1, \ldots, N, \quad dP[a_n = t] = \frac{\lambda(t)}{\Lambda} dt,
\]
To move particle positions \( \xi \) and capacities \( \mu \) for one time step \( \Delta t \) compute
\( (2.13) \)
\[
(a) \quad \mu_m(t + \Delta t) = (1 - r_m)\mu_m(t) + r_m(c_m - \mu_m(t)), \quad r_m = 0 \text{ or } r_m = 1,
(b) \quad P[r_m = 1] = \Delta t\omega_m(\mu_m(t)), \quad P[r_m = 0] = 1 - \Delta t\omega_m(\mu_m(t)),
\]
\[ (c) \xi_n(t + \Delta t) = \xi_n(t) + \Delta t v_n(\vec{\xi}(t), \vec{\mu}(t)), \]

\[ (d) v_n(\vec{\xi}(t), \vec{\mu}(t)) = \begin{cases} 
V(0) & \xi_n < 0 \\
\frac{\mu(\xi_n)}{\Delta y}(\xi_k - \xi_n) & \xi_n \geq 0
\end{cases}, \]

\[ (e) \mu(\xi_n, t) = \sum_m \chi_m(\xi_n) \mu_m(t) \]

**Remark:** We assume in (2.12), for simplicity, that all the processors in the beginning are on.

### 3. The Evolution of the Expectation

This section is devoted to deriving the main result (1.4) of this paper from the particle model (2.12)-(2.13). There are three steps involved. In Section 3.1 we derive a high dimensional Boltzmann-type equation for the joint probability density of the particle positions \( \vec{\xi} \) and the capacity variables \( \vec{\mu} \) of the previous section. In Section 3.2, we then reduce the dimensionality of the problem by employing a type of mean field theory for the conditional probability of the particle positions for a given realization of the capacities. In Section 3.3 we compute averages over time scales which are much longer than the mean on/off switching times \( \tau_{up}^m \) and \( \tau_{down}^m \). The result of this procedure is in zero’th order an evolution equation for the probability \( p(x, t) \) that an arbitrary particle in (2.13) is at position \( x \) at time \( t \) which, up to a multiplicative constant, can be identified with the expectation \( \langle \rho \rangle \) in (1.4).

#### 3.1. The Probability Distribution

We now derive the evolution equation for the probability distribution

\[ F(X, Z, t) dXZ = dP[\vec{\xi}(t) = X, \vec{\mu}(t) = Z] \]

that the particle ensemble \( \vec{\xi} = (\xi_1, \ldots, \xi_N) \) is at the \( N \)-dimensional position \( X = (x_1, \ldots, x_N) \), while the processor capacities \( \vec{\mu} = (\mu_1, \ldots, \mu_m) \) are in the state \( Z = (z_1, \ldots, z_m) \). We have

**Theorem 3.1.** Let the evolution of particles \( \vec{\xi} \) and capacities \( \vec{\mu} \) be given by (2.12) and (2.13). Then, in the limit \( \Delta t \to 0 \) the joint probability distribution \( F(X, Z, t) \) satisfies the initial value problem for the Boltzmann equation

\[ (3.1) \quad (a) \quad \partial_t F + \sum_n \partial_{x_n}[v_n(X, Z, t)F] = \int Q(Z, Z')F(X, Z', t) dZ' \]

\[ (b) \quad F(X, Z, 0) = (AV(0))^{-N}[\prod_{n=1}^N \lambda(-\frac{x_n}{V(0)})][\prod_{m=0}^{M-1} \delta(z_m - c_m)] \]

with the integral kernel \( Q \) given by

\[ (3.2) \quad (a) \quad Q(Z, Z') = \sum_m q_m(z_m, z'_m) \prod_{k \neq m} \delta(z'_k - z_k), \]
Taylor expansion. It is deferred to the Appendix in Section 5.

of choosing the random variables defined by

the Boltzmann equation probability densities for the capacity variables in independence (see c.f. [4]) has to be slightly modified. We first define the probability in (3.1) depends on the capacity variables has a solution of the form evolves independently of each other. This can be seen by the fact that becomes

devises (3.6) implies that every fixed becomes

Note, that the definition (3.6) implies that \( F^c(X, Z, t) dX \) is a probability measure for every fixed \( Z \), i.e. \( \int F^c(X, Z, t) dX = 1 \), \( \forall Z \) holds. Using the definition of \( F^c \) (3.1) becomes

(a) \( \partial_t G = \int Q(Z, Z') G(Z', t) \, dZ' \), (b) \( G(Z, 0) = \prod_{m=0}^{M-1} \delta(z_m - c_m) \)

for \( G \). Note that we obtain a closed equation for \( G \), which is an expression of the fact that the capacities \( \mu^m \) evolve independently of the particles. Moreover, the individual capacities \( \mu_m \) evolve independently of each other. This can be seen by the fact that (3.4) has a solution of the form \( G(Z, t) = \prod_{m=0}^{M-1} g_m(z_m, t) \), where the individual probability densities \( g_m(z_m, t) \), for the state \( z_m \) of the processor \( m \) at time \( t \), satisfy the Boltzmann equation

(b) \( q_m(z_m, z'_m) = \omega_m(z'_m)[\delta(c_m - z'_m - z_m) - \delta(z'_m - z_m)] \).

Proof: The proof of Theorem 3.1 consists of summing up over all the possibilities of choosing the random variables \( r_m \) in (2.13) and an exercise in multidimensional Taylor expansion. It is deferred to the Appendix in Section 5.

In terms of the probability distribution \( F(X, Z, t) \), the expectation \( \langle \rho(x, t) \rangle \) in (1.4) is then given by

\[
\langle \rho(x, t) \rangle = \Delta y \sum_{n=1}^{N} \int \delta(x - \xi_n(t)) F(\xi, Z, t) \, d\xi \, dZ = \Delta y \sum_{n=1}^{N} p_n(x, t),
\]

with \( p_n(x, t) = \int F(\xi_1, \ldots, \xi_{n-1}, x, \xi_{n+1}, \ldots, \xi_N, Z, t) \, d\xi_1, \ldots, \xi_{n-1}, \xi_{n+1}, \ldots, \xi_N \, dZ \) being the probability density that particle number \( n \) is at position \( x \) at time \( t \). The density \( F(X, Z, t) \) is of course too high dimensional to be of practical use, and the goal of the next two subsections is therefore to reduce the dimensionality of the problem.

3.2. Molecular chaos and mean field theory. Since the joint probability \( F \) in (3.1) depends on the capacity variables \( Z \) as well, the usual assumption of statistical independence (see c.f. [4]) has to be slightly modified. We first define the probability for the capacity variables \( Z \) as \( G(Z, t) = \int F(X, Z, t) \, dX \). Integrating out \( X \) in (3.1) gives the initial value problem

\[
\partial_t G = \int Q(Z, Z') G(Z', t) \, dZ', \quad G(Z, 0) = \prod_{m=0}^{M-1} \delta(z_m - c_m)
\]

for \( G \). Note that we obtain a closed equation for \( G \), which is an expression of the fact that the capacities \( \mu^m \) evolve independently of the particles. Moreover, the individual capacities \( \mu_m \) evolve independently of each other. This can be seen by the fact that (3.4) has a solution of the form \( G(Z, t) = \prod_{m=0}^{M-1} g_m(z_m, t) \), where the individual probability densities \( g_m(z_m, t) \), for the state \( z_m \) of the processor \( m \) at time \( t \), satisfy the Boltzmann equation

\[
\partial_t g_m(z, t) = \int q_m(z, z') g_m(z', t) \, dz'
\]

with the kernels \( q_m \) given by (3.2)(b). We now define the conditional probability density \( F^c(X, Z, t) dX = dP[\xi = X | \mu^c = Z] \), which is the probability of the particle ensemble \( \xi \) for a given realization of the \( \mu^c \). The conditional probability density is defined by

\[
F^c(X, Z, t) = \frac{F(X, Z, t)}{G(Z, t)}, \quad G(Z, t) = \int F(X, Z, t) \, dX
\]

Note, that the definition (3.6) implies that \( F^c(X, Z, t) dX \) is a probability measure for every fixed \( Z \), i.e. \( \int F^c(X, Z, t) dX = 1 \), \( \forall Z \) holds. Using the definition of \( F^c \) (3.1) becomes

\[
\partial_t [G F^c] + \sum_n \partial z_n [v_n(X, Z, t) G F^c] = \int Q(Z, Z') F^c(X, Z', t) G(Z', t) \, dZ'
\]
The standard molecular chaos assumption employed in many particle physics (see c.f. [4]) now takes the form that for a given fixed realization of the $\vec{\mu}$ the different $\xi_n$ are independently and identically distributed, i.e. that

$$F^c(X, Z, t) = \prod_n f^c(x_n, Z, t), \quad \int f^c(x, Z, t) \, dx = 1, \forall Z, t$$

holds. The molecular chaos assumption implies therefore the Ansatz

$$F(X, Z, t) = G(Z, t) \prod_n f^c(x_n, Z, t)$$

for the joint probability $F$ in (3.1). $f^c(x, Z, t)$ is the conditional probability density that any of the identical particles is at position $x$ at time $t$ for a given state $Z$ of the processors. To obtain an evolution equation for $f^c(x, Z, t)$, we integrate (3.7) with respect to the variables $x_2..x_N$ and obtain

$$\partial_t[Gf^c(x_1, Z, t)] + \partial_{x_1}[Gf^c(x_1, Z, t)\int v_1(X, Z) \prod_{n=2}^N f^c(x_n, Z, t) dx_2..x_N] =$$

$$\int dZ'Q(Z, Z')f^c(x_1, Z')G(Z')$$

To close the equation (3.8) we have to compute the average mean field velocity $u(x_1, Z, f^c)$ given by

$$u(x_1, Z, f^c) = \int v_1(X, Z) \prod_{n=2}^N f^c(x_n, Z, t) dx_2..x_N$$

asymptotically for large $N$. To this end, we recall from (2.13)(d) that for $x_1 \in [\gamma_m, \gamma_{m+1})$, the interval corresponding to processor number $m$, the velocity $v_1(X, Z)$ is given by

$$v_1(X, Z) = \min\{V(x_1), \frac{z_m}{\Delta y}(x_k - x_1) : x_k > x_1\} .$$

Theorem 3.2 below gives the asymptotic form of the mean field velocity $u(x_1, Z, f^c)$ in the limit for a large number of independent particles ($N \to \infty$).

**Theorem 3.2.** For a given probability measure $f(x)$ and for given constants $V$ and $z$

$$\lim_{N \to \infty \Delta y \to 0} \int \min\{V, \frac{z}{\Delta y}(x_k - x_1) : x_k > x_1\} \prod_{n=2}^N f(x_n) \, dx_2..x_N = \frac{z}{\Lambda f(x_1)}[1-\exp(-\frac{\Lambda V f(x_1)}{z})]$$

holds where $\Lambda = N\Delta y$ is fixed.

**Proof:** The proof is deferred to the Appendix in Section 5.

Thus, in the $m$-th cell $[\gamma_m, \gamma_{m+1})$, we have that the average mean field velocity is asymptotically given by (3.9), and can therefore be expressed by the piecewise
constant function

\begin{equation}
(a) \quad u(x_1, Z, f^c) = \sum_{m=0}^{M-1} u_m(x_1, z_m, f^c) \chi_{[\gamma_m, \gamma_{m+1})}(x_1),
\end{equation}

\begin{equation}
(b) \quad u_m(x_1, z_m, f^c) = \frac{z_m}{\Lambda f^c} \left[ 1 - \exp\left( -\frac{\Lambda V(x_1)f^c}{z_m} \right) \right].
\end{equation}

Therefore (3.8) reduces under the molecular chaos assumption of many independently distributed particles to the mean field Boltzmann equation

\begin{equation}
\partial_t [G f^c] + \partial_{x_1} [u(x_1, Z, f^c) G f^c] = \int dZ' Q(Z, Z') f^c(x_1, Z') G(Z'),
\end{equation}

with the mean field velocity \( u \) given by (3.10). The molecular chaos Ansatz is compatible with the initial condition (3.1)(b) for the probability density \( F(X, Z, t) \). Using (3.1)(b), we obtain the initial conditions

\begin{equation}
f^c(x_1, Z, 0) = 1 - \Lambda V(0) \lambda \left( -\frac{x_1}{V(0)} \right),
G(Z, 0) = \prod_{m=0}^{M-1} \delta(z_m - c_m)
\end{equation}

for the evolution equations (3.11) and (3.4). Note, that \( G \) still satisfies independently the equation (3.4). This is essential, since it guarantees that \( f^c(x_1, Z, t) dx_1 \) is a probability measure, i.e. \( \int f^c(x_1, Z, t) dx_1 = 1, \forall Z, t \) holds.

The probability density \( p_n(x, t) \) of particle number \( n \) being at position \( x \) at time \( t \) in (3.3) is, under the molecular chaos assumption, of the form

\[ p_n(x, t) = p(x, t) = \int f^c(x, Z, t) G(Z, t) \, dZ \quad \forall n. \]

Since all the \( p_n \) are now identical, the expectation \( \langle \rho \rangle \) is, according to (3.3) given by

\[ \langle \rho(x, t) \rangle = \Lambda p(x, t). \]

So, the expectation \( \langle \rho \rangle \) can be identified with the probability \( p \) up to the multiplicative constant \( \Lambda \), giving the total mass in the system. To obtain the evolution equation (1.4) for the expectation \( \langle \rho \rangle \) and the probability density \( p \), we still have to average out somehow the dependence of the conditional probability density \( f^c \) on the processor state variables \( Z = (z_0, \ldots, z_{M-1}) \). The evolution equation for \( p \) - and also for \( \langle \rho \rangle \) - is obtained by integrating out the \( Z \)- variable in (3.11). This gives

\begin{equation}
\partial_t p(x, t) + \partial_{x_1} [p U(x, t, p)] = 0, \quad p U(x, t, p) = \int u(x, Z, f^c) G(Z, t) f^c(x, Z, t) \, dZ.
\end{equation}

Because of the initial condition (3.12) for the conditional probability density \( f^c \) the conservation law (3.13) is subject to the initial condition

\begin{equation}
p(x, 0) = \frac{1}{\Lambda V(0)} \lambda \left( -\frac{x}{V(0)} \right).
\end{equation}

Note, that \( f^c \) still depends on all the capacity variables \( Z \). Therefore (3.13) has to be closed somehow by expressing \( f^c \) in terms of \( p \). In Section 3.3 this closure is achieved by considering a large time regime.
3.3. The large time regime. First, we note that in the setting of Section 2.2 the capacities $\mu_m$ can only assume two discrete values, namely $\mu_m = 0$ or $\mu_m = c_m$. Therefore the probabilities $g_m(z_m, t)$ in (3.5) are concentrated on these values, and we have an exact solution of (3.5) given by

$$g_m(z, t) = g^0_m(t)\delta(z) + g^1_m(t)\delta(z - c_m)$$

Inserting this into (3.5) and using the form of the integral kernels $q_m$ in (3.2)(b) gives

$$\delta(z)\partial_t g^0_m(t) + \delta(z - c_m)\partial_t g^1_m(t) = q_m(z, 0)g^0_m(t) + q_m(z, c_m)g^1_m(t)$$

$$= [-\omega_m(0)\delta(z) + \omega_m(0)\delta(c_m - z)]g^0_m(t) + [-\omega_m(c_m)\delta(c_m - z) + \omega_m(c_m)\delta(z)]g^1_m(t)$$

Comparing the coefficients of $\delta(z)$ and $\delta(z - c_m)$, we obtain that $g^0_m(t)$ and $g^1_m(t)$ are given as solutions of the ODE system

$$\partial_t g^0_m = -\omega_m(0)g^0_m(t) + \omega_m(c_m)g^1_m(t), \quad \partial_t g^1_m = \omega_m(0)g^0_m(t) - \omega_m(c_m)g^1_m(t),$$

which preserves the property of $g_m(z, t)dz$ being a probability measure, i.e. $g^0_m + g^1_m = 1$, $\forall t$ holds. We now consider a regime where the on/off switches of the processors happen very frequently compared to the over time scale, i.e. $\frac{\tau^{\text{up/down}}}{\tau_m} << 1$, $\omega_m = \frac{1}{\tau_m} >> 1$. Thus we rescale the mean up and down times $\frac{\tau^{\text{up/down}}}{\tau_m}$ as well as the frequencies $\omega_m$ in (3.2) by $\frac{\tau^{\text{up/down}}}{\tau_m} \rightarrow \varepsilon \frac{\tau^{\text{up/down}}}{\tau_m}$ and $\omega_m \rightarrow \frac{1}{\varepsilon}\omega_m$. Rescaling the collision kernel $Q$ in (3.11) correspondingly gives the system

$$\partial_t [G f^c] + \partial_z [u(x, Z, f^c)G f^c] = \frac{1}{\varepsilon} \int Q(Z, Z') f^c(x, Z', t) G(Z', t) dZ',$$

with the rescaled collision kernel $Q$ given, according to (3.2), by

$$(a) \quad Q(Z, Z') = \sum_m q_m(z_m, z_m') \prod_{k \neq m} \delta(z_k' - z_k),$$

$$(b) \quad q_m(z_m, z_m') = \omega_m(z_m')[\delta(c_m - z_m') - \delta(z_m' - z_m)].$$

From the above derivation, we have that the probability density $G(Z, t)$ of the processor status factors into $M$ independent densities, supported on $z_m = 0$ and $z_m = c_m$, satisfying the rescaled version of (3.15). Thus we have

$$(a) \quad G(Z, t) = \prod_{m=1}^{M} g_m(z_m, t), \quad g_m(z, t) = g^0_m(t)\delta(z) + g^1_m(t)\delta(z - c_m)$$

$$(b) \quad \varepsilon \partial_t g^0_m = -\omega_m(0)g^0_m(t) + \omega_m(c_m)g^1_m(t), \quad \varepsilon \partial_t g^1_m = \omega_m(0)g^0_m(t) - \omega_m(c_m)g^1_m(t),$$

where the small parameter $\varepsilon$ denotes the ratio of $\tau^{\text{up/down}}$ to the over time scale. The ODE system (3.18)(b) has two distinct eigenvalues, namely zero and $-\frac{\omega_m(0)}{2\varepsilon\omega_m(c_m)}$. This, together with the condition that $g^0_m + g^1_m = 1$ $\forall t$ holds, implies that the $g^0_m$ will converge exponentially on an $O(\varepsilon^2)$ time scale towards their steady state

$$g^0_m(\infty) = \frac{\omega_m(c_m)}{\omega_m(0) + \omega_m(c_m)}, \quad g^1_m(\infty) = \frac{\omega_m(0)}{\omega_m(0) + \omega_m(c_m)}.$$
Therefore, we can, up to exponentially small terms, replace $G(Z, t)$ by $G(Z, \infty)$ in (3.16). Note that $G(Z, \infty)$ is a steady state of (3.4), and therefore satisfies
\begin{equation}
\int dZ' \, Q(Z, Z') G(Z', \infty) = 0, \quad \forall Z .
\end{equation}
Replacing $G(Z, t)$ by $G(Z, \infty)$ in (3.16) we obtain
\begin{equation}
(3.21)
\begin{align*}
(a) \quad & G(Z, \infty) \{ \partial_t [f^c] + \partial_z [u(x, Z, f^c) f^c] \} = \frac{1}{\varepsilon} Q_G[f^c], \\
(b) \quad & Q_G[f^c] = \int dZ' Q(Z, Z') f^c(x, Z') G(Z', \infty) .
\end{align*}
\end{equation}
Expanding the conditional probability density $f^c$ formally in powers of $\varepsilon$ gives that, in zero’th order, $Q_G[f^c] = 0$ holds. Note that, because of (3.20), functions $f^c$ which are independent of $Z$ are automatically in the kernel of the collision operator $Q_G$ in (3.21). Theorem 3.3 states that the kernel of the collision operator consists essentially of only such functions.

**Theorem 3.3.** Any element of the kernel of the collision operator $Q_G$ defined in (3.21)(b) is constant on the vertices of the hypercube $\prod_{m=0}^{M-1} [0, c_m]$. So $Q_G[f] = 0$ implies that
\begin{equation}
(3.22)
\begin{align*}
f(z_1..c_m..z_M) - f(z_1..0..z_M) = 0, \quad \forall m, \quad \forall Z = (z_0, ..., z_{M-1}) \in \prod_m \{0, c_m\}
\end{align*}
\end{equation}
holds.
**Proof:** The proof is deferred to the Appendix in Section 5.

Theorem 3.3 allows us to compute the macroscopic velocity $U(x, t, p)$ in the evolution equation for the probability $p$ in (3.13) in terms of $p$ itself. Since in zero’th order $Q_G[f^c] = 0$ has to hold, $f^c(x, Z, t)$ has to be constant on the hypercube vertices $Z \in \prod_{m=0}^{M-1} [0, c_m]$. In (3.13) we have to compute the flux as
\begin{equation}
(3.23)
\begin{align*}
(a) \quad & p U(x, t, p) = \int u(x, Z, f^c) G(Z, \infty) f^c(x, Z, t) \, dZ, \\
(b) \quad & p(x, t) = \int f^c(x, Z, t) \, dZ .
\end{align*}
\end{equation}
Because of (3.18) $G(Z, \infty)$ is concentrated on the vertices $\prod_{m=0}^{M-1} \{0, c_m\}$ where $f^c$ is constant. Therefore the integral in (3.23)(a) factors, and we obtain
\begin{equation}
(3.24)
\begin{align*}
U(x, t, p) = U(x, p) = \int u(x, Z, p) G(Z, \infty) \, dZ .
\end{align*}
\end{equation}
The derivation above actually computes the zero order term in a Chapman - Enskog procedure for the Boltzmann equation (3.21). The next term would produce a diffusive $O(\varepsilon)$ correction. However, this diffusive correction represents really only a small correction since the mean velocity $U$ of the zero order term is non - zero, i.e. we are still in a primarily hyperbolic instead of a diffusive regime. We now have, in the large time limit, closed the equation (3.13) for the probability density $p(x, t)$ that any of the identical particles is at position $x$ at time $t$. Since this density is up to the
multiplicative constant \( \Lambda \) identical to the expectation \( \langle \rho \rangle \), i.e. \( \langle \rho \rangle = \Lambda p \) holds, we also obtain a closed form equation for the expectation. This equation is of the form

\[
\frac{\partial}{\partial t} \langle \rho(x,t) \rangle + \frac{\partial}{\partial x} [\langle \rho(x) \rangle U(x, \frac{\langle \rho \rangle}{\Lambda})] = 0
\]

Using the initial condition (3.14) for the probability density \( p \), we obtain that the conservation law (3.25) is subject to the initial condition

\[
\langle \rho(x,0) \rangle = \frac{1}{V(0)} \lambda \left( - \frac{x}{V(0)} \right)
\]

We have now assembled all the ingredients for the main result announced in (1.4). Computing \( U(x,p) \) in (3.24), using the form (3.10) of the mean field velocity \( n(x,Z,f^c) \) we have that in the interval \([\gamma_m, \gamma_{m+1}]\), corresponding to the \( m \)-th processor

\[
U(x,p) = \int \frac{z_m}{\Delta p} [1 - \exp(-\frac{\Delta p V(x)}{z_m})] G(Z,\infty) \, dZ, \quad x \in [\gamma_m, \gamma_{m+1}]
\]

holds. Using the fact that \( G(Z,\infty) \) factors into a product of the \( g_m \), and integrating out all variables, except \( z_m \), we obtain

\[
U(x,p) = \int \frac{z_m}{\Delta p} [1 - \exp(-\frac{\Delta p V(x)}{z_m})] g_m(z_m,\infty) \, dz_m, \quad x \in [\gamma_m, \gamma_{m+1}]
\]

Using the formulas (3.18) and (3.19) for \( g_m(z,\infty) \) we integrate with respect to \( z_m \) and replace \( p \) by \( \frac{\langle \rho \rangle}{\Lambda} \), obtaining for the velocity \( U(x,\frac{\langle \rho \rangle}{\Lambda}) \) in (3.25)

\[
U(x,\frac{\langle \rho \rangle}{\Lambda}) = \frac{\omega_m(0)}{\omega_m(0) + \omega_m(1)(\frac{\langle \rho \rangle}{\Lambda})} [1 - \exp(-\frac{\langle \rho \rangle V(x)}{\omega_m})] \text{ for } x \in [\gamma_m, \gamma_{m+1}],
\]

which yields the flux function \( F_E = \langle \rho \rangle U(x, \frac{\langle \rho \rangle}{\Lambda}) \) in (1.4)(a), since

\[
\frac{\omega_m(0)}{\omega_m(0) + \omega_m(1)} = \frac{\tau_{\text{up}}^{m}}{\gamma_{m} + \tau_{\text{down}}^{m}}
\]

holds. Note that the ratio \( \frac{\tau_{\text{up}}^{m}}{\gamma_{m} + \tau_{\text{down}}^{m}} \) is not affected by the rescaling of the mean up and down times \( \tau_{\text{up}}^{m}, \tau_{\text{down}}^{m} \) used in this section. Finally, we remove the technicality of formulating the conservation law as a pure initial value problem, which was used solely to keep the total mass constant in time and to define probability densities. For \( x < 0 \) the velocities of the particles defined in Section 2.2 is constantly equal to \( V(0) \), and therefore also \( U(x,p) = V(0) \) for \( x < 0 \) will hold. The resulting one way wave equation can be solved exactly and, using the initial condition (3.26) we have

\[
\langle \rho(x,t) \rangle = \langle \rho(x - tV(0),0) \rangle = \frac{1}{V(0)} \lambda (t - \frac{x}{V(0)}) \text{ for } x < 0.
\]

Because of flux continuity

\[
F_E|_{x=0^+} = \langle \rho \rangle U(x, \frac{\langle \rho \rangle}{\Lambda})|_{x=0^+} = V(0) \langle \rho(0^-,t) \rangle = \lambda (t)
\]

has to hold, which yields the boundary condition (1.4)(b). The boundary condition (1.4)(b) has to be interpreted in the following way. The flux function at influx \( F_E|_{x=0^+} = \langle \rho \rangle U|_{x=0^+} \) is, because of (3.27), bounded from above by the quantity
If the influx $\lambda(t)$ exceeds this value, as is possible in the transient regime, this results in a flux discontinuity, and correspondingly in a $\delta-$function concentration of the expected density $\langle \rho \rangle$ at the influx boundary at $x = 0$.

Remarks:
- (1.4)(a) says that the whole flux (and not just the capacity $c_m$) is multiplied with the effective up - time $\frac{\tau_{up}^m \tau_{down}^m}{\tau_{up}^m + \tau_{down}^m}$ of processor $m$. This is reasonable, since even for an empty system $\langle \rho \rangle << 1$, the flow will be slowed by shutting down the processors.
- A somewhat puzzling fact is that (1.4) does not reduce to the deterministic conservation law (1.1) in the limit $\tau_{down} \to 0$, i.e. in the case when the processors are always on. The explanation for this is that the derivation of (1.4) is based on the assumption of molecular chaos for the individual particles, and that this assumption is apparently not valid for the deterministic system.

4. NUMERICAL EXPERIMENTS. We now turn to verifying the validity of the approximate conservation law (1.4) for the expectation $\langle \rho(x, t) \rangle$ numerically. We do so by comparing the average over realizations of the numerical solution of (1.1) to the solution of (1.4). So, we first generate $M$ random signals $\mu_m(t)$, $\mu = 0, \ldots, M - 1$ as depicted in Figure 1.1, and compute the corresponding time dependent capacity function $\mu(x, t)$ according to (1.3). For a given realization of $\mu(x, t)$ the conservation law (1.1) is then solved by a standard Godunov method (see [3] for details and c.f. [12] for details on the Godunov method). This process is repeated many times for different realization and one approximation to the expectation $\langle \rho(x, t) \rangle$ is obtained by calculating averages over different realizations. We compare this approximation to the direct solution of the conservation law (1.4), also obtained by a Godunov method. It should be pointed out that the Godunov method reduces to simple upwinding in all cases, since the velocities stay always non - negative. We only employ a first order Godunov scheme since the individual realizations will develop $\delta-$function concentrations as soon as the processors are turned off an $\mu$ becomes zero in certain intervals. The convergence of the first order Godunov method in this case is analyzed and documented in [3], whereas the convergence properties of higher order methods are not so obvious. In all the example below we consider a chain of 40 processors ($M = 40$) which are located in the stage interval $x \in [0, 1]$. For simplicity, we assume that all processors have identical throughput times. This allows us to choose a constant velocity $V(x) = 1$ in (1.1) and (1.4), by choosing an appropriate time scale. Thus $T = 1$ is the throughput time of a part through the whole chain if all processors run below capacity and $T = \frac{1}{40}$ is the throughput time of an individual processor in this case. We set $\tau_{up}^m = \tau_{down}^m = \frac{1}{20}, \forall m$. So the processors run on average only half of the time and we are in the large time scale regime of Section 3.3 since $\varepsilon = \frac{1}{20} << 1$ holds.

Experiment 1: In the first experiment we consider $M = 40$ identical processors with a peak capacity $c_m = 2$, $m = 0, \ldots, 39$. Thus, the flux function $F_E$ in (1.4) is bounded by the effective capacity $\frac{\tau_{up} \tau_{down}}{\tau_{up} + \tau_{down}} = 1$. We start with an empty system $\rho(x, 0) = 0$ and use a constant influx $\lambda(t) = 0.5$, well below the effective capacity. Figure 4.1 shows the expectation $\langle \rho(x, t) \rangle$ computed by averaging 200 realizations of (1.1) and by solving (1.4). Note, that we obtain good quantitative agreement in
Fig. 4.1. Experiment 1: Left Panel: Density $\rho$ from the deterministic conservation law (1.1) with random capacities and constant influx $\lambda = 0.5$. Averaged over 200 realizations. Right Panel: Expectation $\langle \rho \rangle$ of the density $\rho$ according to the mean field model (1.4) with constant influx $\lambda = 0.5$.

Fig. 4.2. Experiment 2: Peak capacities $c_m$ for $M = 40$ processors.

the size of the steady state distribution as well as in the transient behavior, i.e. the velocity of the wave propagating from $x = 0$ to $x = 1$. From (1.4) we deduce, that the steady state density $\langle \rho(x, \infty) \rangle$ is given by the equation $F \left( \frac{1}{2}, 2, 1, \langle \rho(x, \infty) \rangle \right) = \lambda$ or $1 - e^{-\langle \rho(x, \infty) \rangle/2} = 0.5$.

**Experiment 2:** In the second experiment we keep the setup of the first experiment, but introduce a bottleneck in processors 11 – 20. The peak capacities $c_m$ are shown in Figure 4.2. Note that, because of our choice of mean up and down times $\tau_{up}^m$ and $\tau_{down}^m$, the effective capacities are half of the peak capacities shown in Figure 4.2. We choose a constant influx $\lambda = 0.27$. Again, Figure 4.3 shows the expectation $\langle \rho(x, t) \rangle$ computed by averaging 200 realizations of (1.1) and by solving (1.4).

**Experiment 3:** The promise of conservation law models for supply chains lies in their ability to provide a relatively inexpensive way to model the transient behavior of supply chains far from steady state regimes. Therefore, we perform the third experiment for a regime which is truly far from equilibrium. We keep the setup from the second experiment but use a transient influx density, shown in Figure 4.4. Note, that the initial influx density $\lambda = 0.7$ is below the effective capacity $\frac{c_m \tau_{up}^m}{\tau_{down}^m + \tau_{up}^m}$ for most processors, but exceeds the effective capacity for the bottleneck processors for $m = 11, \ldots, 20$. Beyond $t = 1$ the transient influx $\lambda(t)$ is then well below the effective
Fig. 4.3. Experiment 2: Left Panel: Density $\rho$ from the deterministic conservation law (1.1) with random capacities and constant influx $\lambda = 0.27$, 40 cells, bottleneck in cells 11-20. Averaged over 200 realizations. Right Panel: Expectation $\langle \rho \rangle$ of the density $\rho$ according to the mean field model (1.4) with constant influx $\lambda = 0.27$, 40 cells, bottleneck in cells 11-20.

Fig. 4.4. Experiment 3: Transient influx density.

capacity for all processors. Thus, we will see a wave propagating through the first 10 processors $0 < x < 0.25$, the buildup of queues in the next 10 bottleneck processors $0.25 < x < 0.5$, and relaxation towards steady state after $t = 1$. Figure 4.5 shows the expectation $\langle \rho(x, t) \rangle$ computed by averaging 500 realizations of (1.1) and by solving (1.4). We observe again, that the size of the peaks (the maximal queue length in front of the processors) as well as their location in the $x - t$ plane (the transient response) is given accurately by the mean field model (1.4).

5. APPENDIX. We start out by proving the evolution equation (3.1) for the joint probability density $F(X, Z, t)$ of the particle positions and processor states.

Proof of Theorem 3.1:

Once the random variables $r_m$ in (2.6) are chosen the rest of the evolution is completely deterministic. Summing up over all possible choices of the vector $R = (r_1, ..., r_M)$ and weighting them with their probabilities, gives

$$F(X, Z, t + \Delta t) = \int dX' Z' R F(X', Z', t) \prod_n [\delta(x'_n + \Delta t v_n(X', Z') - x_n)]$$
We formulate the above relation weakly in $X$ by integrating against a test function $\psi(X)$:

\[ \int \psi(X) F(X, Z, t + \Delta t) \, dX = \int dX' F(X', Z, t) \psi(X' + \Delta t V(X', Z')) \]

where the vector $V$ denotes $(v_1, \ldots, v_N)$. We Taylor-expand the terms on the right hand side of (5.1) in $\Delta t$ up to first order, and obtain after some calculus

\[ \int \psi(X) F(X, Z, t + \Delta t) \, dX = \int dX F(X, Z, t)\psi(X) + \Delta t \int dX' F(X', Z, t) V(X', Z) \cdot \nabla_X \psi(X') + \Delta t \int dX' F(X', Z, t) \psi(X) Q(Z, Z') \]

with the integral kernel $Q$ given by (3.2). Letting $\Delta t \to 0$ this gives the weak form of

\[ \partial_t F + \sum_n \partial_{x_n} [v_n(X, Z, t) F] = \int dZ' Q(Z, Z') F(X, Z') \]

We now proceed to prove the form of the mean field velocity $u(x_1, Z, f_c)$ in (3.10), i.e. we prove Theorem 3.2. To prove Theorem 3.2 we will need the following auxiliary lemma, giving the expectation of the minimum of $m$ independent random numbers, which are equidistributed in the interval $[0, 1]$. 

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**Fig. 4.5.** Experiment 3: Left Panel: Density $\rho$ from the deterministic conservation law (1.1) with random capacities and transient influx, 40 cells, bottleneck in cells 11-20. Averaged over 500 realizations. Right Panel: Expectation $\langle \rho \rangle$ of the density $\rho$ according to the mean field model (1.4) with transient influx, 40 cells, bottleneck in cells 11-20.
In order to prove (3.9), we have to compute the limit of the quantity

$$E_m = \int \min_{k=1}^m \chi_{(0,1)}(\omega_k) \, d\omega_1 \ldots d\omega_m = \frac{1}{m+1}, \quad m = 1, 2, \ldots,$$

where $\chi_{(0,1)}$ denotes the usual indicator function on the interval $[0, 1]$.

**Proof:** The proof is based on induction in $m$. We denote by $R_m(s)$ the antiderivative of the probability density function of the random variables $\omega_1, \ldots, \omega_m$, uniformly distributed in $[0, 1]$. So we have

$$R_m(s) = d\mathcal{P}[\min\{\omega_k : k = 1, \ldots, m\} = s], \quad R_m(0) = 0, \quad R_m(1) = 1,$$

and derive a formula for $R_m$ recursively. The derivative $R'_m(s)$ is given by

$$R'_m(s) = \int_{[0,1]^m} \delta(s - \min\{\omega_1, \ldots, \omega_m\}) d\omega_1 \ldots d\omega_m = \int_{[0,1]^m} \delta(s - \min\{\omega_1, \ldots, \omega_{m-1}, \omega_m\}) d\omega_1 \ldots d\omega_m =$$

$$\int_{[0,1]^2} \delta(s - \min\{r, \omega_m\}) R'_{m-1}(r) \, dr d\omega_m = \int_{[0,1]^2} (H(r - \omega_m) \delta(s - \omega_m) + H(\omega_m - r) \delta(s - r)) R'_{m-1}(r) \, dr d\omega_m$$

Computing these integrals gives, because of $R_{m-1}(1) = 1$, the recursion

$$R'_m(s) = 1 - R_{m-1}(s) + (1 - s) R'_{m-1}(s) = \frac{d}{ds} [s + (1 - s) R_{m-1}(s)]$$

and because of $R_m(0) = 0$, $\forall m$ we obtain the recursive formula

$$R_m(s) = s + (1 - s) R_{m-1}(s), \quad R_1(s) = s \quad (5.3)$$

Solving the recursion (5.3) via induction gives $R_m(s) = 1 - (1 - s)^m, \quad m = 1, 2, \ldots$ The expectation $E_m$ is now given by

$$E_m = \int_0^1 s R'_m(s) \, ds = 1 - \int_0^1 R_m(s) \, ds = \int_0^1 (1 - s)^m \, ds = \frac{1}{m+1}$$

With the aid of the above lemma we are able to prove the mean field result of Theorem 3.2:

**Proof of Theorem 3.2:**

In order to prove (3.9), we have to compute the limit of the quantity

$$u(x_1, z, V, f) = \int \min\{V, \frac{z}{\Delta y} (x_k - x_1), \quad x_k > x_1, \quad k = 2 \ldots N\} \prod_{n=2}^N f(x_n) \, dx_2 \ldots x_n$$
as $N \to \infty$, $\Delta y \to 0$ with $N\Delta y = \Lambda$ remaining constant, for a given probability measure $f$ and constants $z$ and $V$. $u$ can be interpreted as the expectation of the quantity $\min\{V, \frac{\Delta y}{z}(x_k - x_1), \ x_k > x_1, \ k = 2, N\}$, where $x_2, \ldots, x_N$ are random variables independently and identically distributed according to the measure $f(x)$. We note that the variable $x_k$ only contributes to the minimum if $x_1 < x_k < x_1 + \Delta y$ holds. For any $k = 2, \ldots, N$ let $\Delta y\tilde{p}$ denote the probability that $x_k \in (x_1, x_1 + \frac{\Delta y}{z})$ holds. Clearly, $\tilde{p}$ is given by

$$\tilde{p} = \frac{1}{\Delta y} \int_{x_1}^{x_1 + \frac{\Delta y}{z}} f(s) \ ds = \frac{V f(x_1)}{z} + O(\Delta y),$$

and the probability that none of the $x_k, \ k = 2, \ldots, N$ is in the interval, i.e. the probability that $u = V$ holds is given by $(1 - \Delta y\tilde{p})^{N-1}$. We now compute the probability $p_m$ that of the $N-1$ variables $x_2, \ldots, x_N$ precisely $m \geq 1$ lie in the interval. $p_m$ is given by

$$p_m = \binom{N-1}{m} (\Delta y\tilde{p})^m (1 - \Delta y\tilde{p})^{N-1-m},$$

where the binomial coefficient denotes the number of possible ways to choose $m$ variables and the other terms denote the probabilities that, for such a choice, the chosen $m$ variables lie in the interval and the others do not. In the case that precisely $m$ variables lie in the interval, their probability distribution can be replaced by the conditional probability, given that we already know that they are in the interval. This conditional probability is given by

$$q(s) ds = dP[x_k = s \mid x_k \in (x_1, x_1 + \frac{\Delta y}{z})]$$

or

$$q(s) = \chi_{[x_1, x_1 + \frac{\Delta y}{z}]}(s) \frac{f(s)}{\Delta y\tilde{p}}$$

Thus we obtain

$$u(x_1, z, V, f) = p_0 V + \sum_{m=1}^{N-1} p_m \frac{z}{\Delta y} \int \min\{s_k - x_1 : k = 1, \ldots, m\} \prod_{k=1}^{m} q(s_k) \ ds_1 \ldots s_m$$

substituting $s_k = x_1 + \frac{\Delta y}{z}\omega_k$ in the integral gives

$$u(x_1, z, V, f) = p_0 V + \sum_{m=1}^{N-1} p_m V \int \min\{\omega_k : k = 1, \ldots, m\} \prod_{k=1}^{m} \left(\frac{\Delta y}{z} q(x_1 + \frac{\Delta y}{z}\omega_k)\right) d\omega_1 \ldots \omega_m$$

computing the probability density according to (5.5) gives

$$\frac{\Delta y}{z} q(x_1 + \frac{\Delta y}{z}\omega_k) = \frac{V}{z} \chi_{[0,1]}(\omega_k) \frac{f(x_1 + \frac{\Delta y}{z}\omega_k)}{\tilde{p}} = \chi_{[0,1]}(\omega_k) + O(\Delta y)$$

Thus the $\omega_k$ are up to order $O(\Delta y)$ uniformly distributed in $[0, 1]$, and we have

$$u(x_1, z, V, f) = p_0 V + \sum_{m=1}^{N-1} p_m V[E_m + O(\Delta y)]$$
with the integral $E_m$ given by

$$E_m = \int \min\{\omega_k : k = 1, \ldots, m\} \prod_{k=1}^m \chi_{(0,1)}(\omega_k) \, d\omega_1 \ldots d\omega_m$$

$E_m$ is the expectation of the minimum of $m$ uniformly distributed random variables and, according to the auxiliary lemma (5.2), $E_m = \frac{1}{m+1}$ holds.

Using this result in (5.6) gives

$$u(x_1, z, V, f) = V \sum_{m=0}^{N-1} p_m \left[ \frac{1}{m+1} + O(\Delta y) \right]$$

Because of (5.4) we have that $\sum_{m=0}^{N-1} p_m = 1$ holds. Therefore, the $O(\Delta y)$ term can be neglected, although it appears inside the summation, and we have

(5.7) $$u(x_1, z, V, f) = V \sum_{m=0}^{N-1} \frac{1}{m+1} \left( \binom{N-1}{m} N^{N-1-m} \right) (1 - \Delta y \bar{p})^N$$

A simple application of the binomial theorem yields that

$$\sum_{m=0}^{N-1} \frac{1}{m+1} \left( \binom{N-1}{m} N^{N-1-m} \right) = \frac{(a + b)^N - b^N}{Na}, \quad \forall a, b$$

holds. With the obvious choice of $a$ and $b$ we obtain from (5.7)

$$u(x_1, z, V, f) = V \frac{1 - (1 - \Delta y \bar{p})^N}{N\Delta y \bar{p}} + O(\Delta y) = V \frac{1 - e^{-\Lambda \bar{p}}}{\Lambda \bar{p}} + O(\Delta y) \ .$$

(Remember $\Lambda = N\Delta y = \text{const}$ holds!) together with $\bar{p} = \frac{Vf(x_1)}{2} + O(\Delta y)$ this gives (3.9).

Finally, we prove the structure of the kernel of the collision operator $Q_G$ in Theorem 3.3.

**Proof of Theorem 3.3**

From (3.17) we have that the collision kernel $Q$ of the operator $Q_G$ is of the form

$$Q(Z, Z') = \sum_m q_m(z_m, z'_m) \prod_{k \neq m} \delta(z_k - z'_k), \quad q_m(z_m, z'_m) = \omega_m(z'_m) [\delta(c_m - z_m - z'_m) - \delta(z'_m - z_m)]$$

At the same time, we have from (3.18) that the steady state $G(Z, \infty)$ of the processor state distribution is supported only on the hypercube $\prod_{m=0}^{M-1} \{0, c_m\}$. So $G(Z, \infty)$ is of the form

$$G(Z, \infty) = \prod_m g_m(z_m), \quad g_m(z_m) = g_m^0 \delta(z_m) + g_m^1 \delta(z_m - c_m) \ .$$

Inserting this into the definition (3.21)(b) of the collision operator $Q_G$ gives

$$Q_G[f] = \sum_m \int q_m(z_m, z'_m) [g_m^0 \delta(z'_m) + g_m^1 \delta(z'_m - c_m)] f(Z') \prod_{k \neq m} \delta(z_k - z'_k) g_k(z'_k) \, dZ'$$
Integrating out all variables except $z'_m$ in each term of the sum above yields

$$Q_G[f] = \sum_m q_m(z_m, z'_m) \left[ g_m^0 \delta(z'_m) + g_m^1 \delta(z'_m - c_m) \right] f(z_1...z'_m...z_M) \prod_{k \neq m} g_k(z_k)$$

$$= \sum_m [q_m(z_m, 0) g_m^0 f(z_1...0...z_M) + q_m(z_m, c_m) g_m^1 f(z_1...c_m...z_M)] \prod_{k \neq m} g_k(z_k).$$

Using the form (3.17)(b) of the individual kernels $q_m$ gives

$$Q_G[f] = \sum_m \left( \omega_m(0) \delta(c_m - z_m) - \delta(z_m) \right) [g_m^0 f(z_1...0...z_M) + \omega_m(c_m) \delta(z_m) - \delta(c_m - z_m)] g_m^1 f(z_1...c_m...z_M)] \prod_{k \neq m} g_k(z_k).$$

Using the form (3.19) of the coefficients $g_m^0(\infty)$ and $g_m^1(\infty)$ of the steady distribution, and collecting terms, gives

$$Q_G[f] = \sum_m \frac{\omega_m(0) \omega_m(c_m)}{\omega_m(0) + \omega_m(c_m)} [\delta(c_m - z_m) - \delta(z_m)] [f(z_1...0...z_M) - f(z_1...c_m...z_M)] \prod_{k \neq m} g_k(z_k).$$

Therefore $Q_G[f]$ can vanish identically $\forall Z$ only if

$$f(z_1...c_m...z_M) - f(z_1...0...z_M) = 0, \forall m, \forall Z \in \prod_m \{0, c_m\}$$

holds.

REFERENCES


