A continuum description for a DES control problem

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Abstract—In Perdaen et al. [1] we developed a Discrete Event Simulation (DES) for a re-entrant semiconductor factory with a push dispatch policy at the beginning of the line and a pull dispatch policy at the end of the line. The simulation uses a heuristic to move the transition point between both policies, the push-pull point (PPP), along the production line. The paper showed that the heuristic PPP control together with a CONWIP policy as a starts policy reduces the mismatch between a fluctuating demand and a fluctuating production output. In this paper we develop a continuum model for the production line with dispatch policies using transport equation models. We analyze the resulting discontinuous partial differential equations and show that the movement of the PPPs generate shock waves that travel along the supply chain. The heuristic control approach of the DES is implemented in the continuum setting. We propose a project based on adjoint calculus to determine an optimal control policy for the PDE model.

I. INTRODUCTION

Semiconductor wafers are produced in re-entrant production lines: Wafers return to the same set of machines about 20 to 30 times before exiting the factory. As a result, cycle time through such a factory is of the order of weeks. The production planning problem, i.e. to control the outflux of a factory for such a production setup becomes extremely complicated. While production and demand vary stochastically on a timescale of weeks, the most obvious control which is the number of parts that is started into the factory has an influence on a timescale of months.

A. Discrete Event Simulations for small scale control

In [1] we considered another control actuator that has an influence on the outflux manifesting itself on a much faster timescale: Any machine in a re-entrant production line that produces multiple steps will have to decide which step it is serving next. This is known as the dispatching rule. We generated a Discrete Event Simulation model of a stylized semiconductor factory exhibiting all the important steps of semiconductor production. As dispatch policies we used at the beginning of the line a push policy, also known as first buffer first served. A push policy gives priority to early production steps over later production steps. At the end of the production line we used a pull policy giving priority to higher production steps over lower steps. Pull policies are known as shortest-expected-remaining-process-time policies. The step at which we switch from a push to a pull policy is called the push-pull point. By moving the PPP we can control how much Work in Progress (WIP) is competing for the last few production steps and hence how fast the production moves through those steps. Figure 1 shows the steady state outflux as a function of the WIP levels for a 20 fixed PPP locations as a result of DES experiments. Note that the top curve corresponds to a pure pull policy while the bottom curve corresponds to a pure push policy.

![Fig. 1. Mean outflux as a function of mean WIP in steady state for different positions of the PPP (from [1])]()
the product density is described by the continuity equation:

\[
\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x} (F(\rho(x,t))) = 0.
\]

(1)

Here the space variable \( x \in [0,1] \) denote the stages of production, where \( x = 0 \) represents the beginning of the production line and \( x = 1 \) the end. \( F(\rho) \) is a flux function given as a state equation describing the flux over the whole production line as a function of the WIP in the production. This is known as a clearing function approach [5]. We have argued in [4] that for a strongly re-entrant flow the clearing function can be written as

\[
F(x, t) = \frac{\mu}{1 + W(t)} \rho(x, t)
\]

(2)

where \( \mu \) is the overall maximal production rate and \( W(t) = \int_0^1 \rho(x,t)dx \) is the total WIP in the factory.

To have a complete setup for a transport equation we need boundary and initial conditions: \( \lambda(t) \) is the rate of products entering the factory at \( x = 0 \)

\[
\lambda(t) = v(\rho(x,t)) \rho(x,t) \big|_{x=0}
\]

(3)

and \( \rho(x,t) = \rho_0(x) \) is the initial WIP-distribution. In [2] a control algorithm was presented that controls the outflux of the continuum model solely by regulating the influx (in control theoretic terms this is a tracking problem). Two problems were solved: i) Minimizing the \( L_2 \) norm for the difference of the outflux and a demand rate target over a fixed time period (demand tracking problem) and ii) Minimizing the \( L_2 \) norm for the difference between the total number of parts that have left the factory and the desired total number of parts over a fixed time period (backlog problem).

Mathematically this becomes a constrained control problem where the constraint is generated by the fact that the state variables are solving a hyperbolic PDEs (for related approaches see [6], [7], [8]. The approach is formal and based on the adjoint method of variational optimization which allows to deal with jumps in the initial and boundary values as well as with the non-locality in the flux function. Specifically the cost functional \( J \) measuring the mismatch between a given demand \( d(t) \) and the outflow is minimized:

\[
\min J(\rho, \lambda) = \frac{1}{2} \int_0^T (d(t) - v(\rho(t)))^2 dt \quad \text{subject to (4)}
\]

\[
\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x} (\lambda(t)) = 0 \quad (5)
\]

\[
\lambda(t) = v(\rho(0,t))
\]

\[
\rho_0(x) = \rho(x,0)
\]

\[
v(\rho) = \frac{v_{\text{max}}}{1 + \int_0^1 \rho(x,t)dx}.
\]

(8)

Figure 2 shows the outcome of the tracking problem. Note the strong nonlinearity of the problem: In order to achieve a sinusoidal outflux, we need a sawtooth-like influx.

C. Multiscale control problems

Our long term goal is to develop control algorithms working simultaneously on several time scales. An interesting multiscale control problem comes from merging the dispatch control of [1] with the tracking control of [2]: Control of production systems based on the influx is a large scale control algorithm - long time scales and over many production steps. In [1] we have shown at a simulation level based on DES that dispatch policies can be used to effect small scale control - short time scales and acting on individual machines. The PPP-dispatch control required that we have a CONWIP starts policy in order to make up for the fluctuations in output. However, starts policies are the domain of large scale controls and are used to affect the long term production plans.

The first step for a multiscale control problem is to mimic the PPP control for the DES system in a continuum model of the type in [4] using a control action that moves the PPP instantaneously along the production line. A second step will be to merge the dispatch control with the tracking problem. The integration of both approaches will enable us to control for e.g. seasonal demand swings while at the same time adjust for daily production variation and daily demand. Another scenario involves the transition between ramping down an old product while starting up a new product over the timescale of months while at the same time serving the stochastically varying demand in weekly increments. The fundamental difficulty comes from the fact that, while the tracking problem uses the influx function as the control variable, the dispatch control needs CONWIP, i.e. a starting policy given as a result of the optimization with respect to daily fluctuations. Hence we are faced with a typical dynamic multi-objective optimization problem. We expect to develop a Pareto-optimal algorithm that allows us to understand the trade-offs between satisfying short term demand fluctuations and staying on a long term plan.

II. CONTINUUM MODEL FOR THE A FLOW WITH PUSH-PULL DISPATCH POLICIES

We assume there exists a control variable \( \xi = \xi(t) \in [0,1] \) describing the position of the push–pull point (PPP).
To derive clearing functions depending on $\xi$ we make the following basic assumptions:

- Moving the PPP does not change the production density nor the WIP but it does change the priority rules and hence the production velocity.
- We assume that within the regions of the production line where push (pull) policies apply the velocity stays independent of the production stage. This is an approximation that could be relaxed and made more realistic.

We can then define flux and velocity functions for the two regions:

$$F_{\text{push/pull}}(x,t) = \frac{v(t)_{\text{push/pull}} \rho(x,t)}{1+C_{\text{push/pull}} W(t)}$$

Notice that $\lim_{W \to 0} v_{\text{push/pull}} = v_0$, since if there is no material in the factory there is nothing to dispatch - any item will move with the raw velocity $v_0$ and there are no interactions between items. Since we showed in [1] (cf. Figure 1 that $F_{\text{push}} < F_{\text{pull}}$ we have $C_{\text{push}} > C_{\text{pull}}$.

- We have the following constraints for the overall velocity function $v(t,x,\xi)$:

$$\lim_{\xi \to 1} v(t,x,\xi) = v(t)_{\text{push}}$$
$$\lim_{\xi \to 0} v(t,x,\xi) = v(t)_{\text{pull}}$$
$$v(t,x,\xi) = C_1 \text{ for } x < \xi$$
$$v(t,x,\xi) = C_2 \text{ for } x > \xi, \quad C_1 \leq C_2$$

$$\lim_{W^+ \to 0} v(t,x,\xi) = v_0 \text{ for } x > \xi$$
$$\lim_{W^- \to 0} v(t,x,\xi) = v_0 \forall x.$$  

where $W^+ = \int_{\xi}^{1} \rho(x,t)dx$ is the WIP in the pull region. Notice the difference between equation (11) and (12): The WIP in the pull region influences the velocity in the push region but the WIP in the push region does not influence the velocity in the pull region, since there is a global priority of pull over push.

A parsimonious model for the velocity that satisfied these constraints is given by

$$v(t,x,\xi) = \begin{cases} 
  \frac{v_0}{1+C_{\text{push}} \int_{\xi}^{1} \rho(u,t)du} & \text{for } x < \xi \\
  \frac{v_0}{1+C_{\text{pull}} \int_{\xi}^{1} \rho(u,t)du} & \text{for } x > \xi 
\end{cases}$$

where $v(t,x,\xi) = v_{\text{push}}(t)$ for $x < \xi$; $v(t,x,\xi) = v_{\text{pull}}(t)$ for $x > \xi$. Hence with the velocity equation (13), the flux equation (9), the boundary condition (3) and mass conservation (1) we have a well defined hyperbolic partial differential equation setup.

### III. Analysis of the Model

#### A. Steady states

The first step in analyzing this model is to determine the steady state WIP-distribution: Without loss of generality we scale $v_0 = 1$, $C_{\text{pull}} = 1$ and define $C_{\text{push}} = C$. Then, setting $F_{\text{push}} = F_{\text{pull}} = \lambda$ we find the steady state WIPs

$$W_{\text{pull}} = \frac{\lambda}{1-\lambda(1-C)(1-\xi)}$$
$$W_{\text{push}} = \frac{\lambda(1-\xi)(1-C)\xi}{1-\lambda(1-\xi)(1-C)}$$

Figure 3 shows the steady state WIP profiles for different PPP with $\lambda = 0.7$ and $C = 1.2$. Note that the WIP in the push region is always higher than the one in the pull region and that the jump in the density is located at the PPP. Figure 4 shows that the total WIP has a minimum for a PPP in the middle of the production line. By Little’s law, the overall
production will be fastest when the PPP is at this minimum value.

\textbf{B. Control action}

Assuming the system is in steady state we can now analyze the influence of a control action. Since the location of the PPP is determined by a policy, it can be changed instantaneously relative to the motion of the parts through the factory. Hence, the physical position of any part in the factory does not change but the priority of a part changes. As a result, the velocity changes and with the velocity the flux over the production line changes too. Starting from a steady state with a constant flux over the whole line, then leads to intervals of flux increase and intervals of flux decrease. Moving the PPP from position $\xi_1$ to position $\xi_2$ where $\xi_2 < \xi_1$ brings a new region of high WIP into the pull regime, characterized by high velocity. The result is an increased flux in that interval. For $\xi_2 > \xi_1$ we move a region of low density into the push regime with low velocity leading to a depressed flux. At the same time moving the PPP upstream increases the WIP in the pull region and hence decreases the flux at the end of the production line whereas moving the PPP downstream reduces the WIP competing with each other in the pull region and hence increases the flux at the end.

![Flux distribution along the production line for a move of the PPP from $\xi = 0.2$ to $\xi = 0.7$.](image1)

**Fig. 5.** Flux distribution along the production line for a move of the PPP from $\xi = 0.2$ to $\xi = 0.7$.

Figure 5 and Figure 6 shows the flux profiles after a move downstream or upstream, respectively.

**C. Shock waves**

Changing the PPP introduces discontinuities in the flux at the position of the old PPP and at the position of the new PPP. Since at the old PPP there already was a discontinuity of the WIP-profile, we have a typical situation for the creation of a shock wave [9]. The shock travels with the speed given by the Rankine Hugoniot condition:

$$v_{\text{shock}} = \frac{\Delta F}{\Delta \rho} = \frac{v^{\text{Push/Pull}}\rho^- \Delta \rho^-}{\Delta \rho} = v^{\text{Push/Pull}}.$$

Inspection of Figures 5 and 6 shows that the shock speed is positive in both cases, leading to shock waves that travel downstream. In the case of a move of the PPP upstream, the shockwaves travels to the end of the production line. If the PPP moves downstream the associated shock wave travels to the new PPP and gets stuck there.

For the discontinuity arising at the new PPP we can prove the following proposition:

**Proposition**

Assume that the PPP point at $x = \xi$ and the influx $\lambda$ are constant. Assume furthermore a non-steady WIP distribution given by two constants $\rho^-(x), x < \xi$ and $\rho^+$ for $x > \xi$ such that $\rho^- v^- = \lambda$, and $\rho_0(x) = H(x - \xi)\rho^- + H(\xi - x)\rho^+$. Provided that $v^*$ is strictly positive, then

$$\rho(x, t) = \rho^*(x, t)$$

is a weak solution of the problem for $\xi < x < \xi + V^*(t)$, $t > 0$. Here $\rho^*$ is the solution to the initial–value problem on $(x, t) : \xi \leq x \leq \xi + V^*(t)$ and $t > 0$

$$\partial_t \rho^* + v^*(t) \partial_x \rho^* = 0,$$

$$\rho^*(t, \xi) v^*(t) = \lambda,$$

$$v^*(t) := \frac{\lambda}{1 + C^{\text{Pull}} \int_{\xi}^{\xi + V^*(t)} \rho^*(x, t) dx + \int_{\xi}^{\xi + V^*(t)} \rho^+ dx},$$

$$V^*(t) := \int_0^t v^*(\tau) d\tau.$$

Note that the problem for $\rho^*$ is well–defined using characteristics. The function $v^*$ is strictly positive and therefore
the problem can be rewritten as

$$\partial_x \rho^* + \frac{1}{v^*(t)} \partial_t \rho^* = 0$$

with initial data $\rho^*(x = \xi, t)$ for $t \geq 0$. Furthermore, no boundary conditions are necessary, since the solution $\rho^*$ exists only in the domain bounded by the extremal characteristic $x = \xi + \int_0^t v^*(\tau)d\tau = \xi + V^*(t)$ and $x = \xi$. However, the initial value $\rho^*(t, \xi)$ is only given implicitly since the speed of propagation $v^*$ depends itself on $\rho^*$. Further, one easily checks that $\rho^*$ satisfies along the line $(x = \xi, t), t \geq 0$, the jump condition

$$\rho^* v^* = \rho^- v^{\text{push}} = \lambda.$$  

The possible second discontinuity along $(x, t) : x = \xi + V^*(t), t \geq 0$ satisfies the Rankine–Hugoniot jump condition and travels downstream. The later is trivially satisfied since the governing equation is a linear transport equation. Therefore, $\rho^*$ is a weak solution.

**IV. Outlook**

**A. Numerics**

We use a wave-front-tracking procedure to resolve the dynamics. The wave interactions are given by Rankine-Hugoniot conditions. We assume that initial data $\rho_0(x)$, the inflow $\lambda$ and the evolution of the push–pull point $\xi$ are piecewise constant. We introduce a numerical grid $\{t_n\}_{n \in \mathbb{N}}$ in time such that $t_{n+1} - t_n = \Delta t$. We do not need a spatial grid. The algorithm proceeds as follows. At time level $t^n$ the piecewise constant solution $\rho_n$ for $0 \leq x \leq 1$ is given. The solution consists of $M$ traveling discontinuities located at positions $0 \leq x^i \leq 1$ for $i = 1, \ldots, M$. Each discontinuity separates two densities $\rho_{n-1}, \rho_n$. We compute the transport velocities in the push and pull part of the factory given by Equations 13 where the integrals are evaluated analytically.

Note that the discontinuities within each part of the supply chain cannot interact since the push velocity is always smaller than the pull velocity and both are non-negative by assumption. We evaluate all events, i.e. movements of the PPP, merging of shock waves with stationary shocks at the PPP and exiting of shock waves at the end of the production line. While they possibly lead to new traveling discontinuities, the maximal number of discontinuities grows linear in the number of timesteps and hence stays bounded.

**B. Control**

We will reproduce the DES experiments in [1]. There is an important timescale that was kept constant in the DES: The PPP was moved after a fixed number of days corresponding to roughly half of the mean cycle time. Since changing the PPP has very nonlocal effects, the frequency of change will be very important for the effectiveness of the PPP algorithm to track a stochastically varying outflux. Having a partial differential equation model for the PPP-control allows us much faster simulation experiments. We will therefore explore the influence of varying the update frequency for the PPP-moves. It is obvious that there exists an optimal frequency since waiting a very long time for moving the PPP is not an effective strategy to track relatively high frequency demand variations and adjusting at a very high frequency leads to chattering and no changes in the outflux.

Finally, a PDE model can be used to solve a multi-objective production planning problem based on optimizing a cost functional parametrized by the relative importance of short term demand fluctuation and long term structural demand changes related to seasonality or the introduction of new products or phasing out of old products. Related to short and long term goals are the control actuators that affect the outflux on short and long term scales - dispatch policy via the PPP and starts policies.

**References**


