A KINETIC MODEL FOR AN AGENT BASED MARKET SIMULATION.

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Abstract. This is the abstract of your paper and it should not exceed 200 words.

1. Introduction. Modeling the success and failure of new products or new technologies in the market place has a long tradition of mathematical modeling. One of the most cited papers in that context is [?] which categorized the potential adopters of a product into innovators and followers. Innovators’ decision to buy a product are mostly driven by the properties of the product itself (its utility, its price, its features) whereas followers are in addition strongly motivated by the success of the product, i.e whether others have bought the product too. The resulting Bass model leads to a logistic type of equation for the time evolution of the market share. The derivative of that curve represents the time evolution of the sales rate, typically showing a small starting rate, a single well defined maximum and then an exponentially decaying tail.

The Bass model is one of a large number of aggregate models for the way consumers adopt a new product. These models are typically based on the intuition of the modeler. The standard approach at validation has been to perform statistical fits of the parameters of the aggregate model tying the parameters of the aggregate model characterizing the buying decisions of individuals to the parameters that describe the utility of the product for an individual customer.

An alternative approach to a purely statistical fit is to develop agent based models, whereby agents representing potential buyers are imbued with an individual utility measure whose evaluation in a changing environment will lead to a decision
to buy a product or not. The changing environment may include the amount of information on the product available to the agent, the changing price and relative competitiveness of the product and the relative importance given to this factors over time by an agent. The resulting agent based simulation then represents such decisions processes for a large number of agents over a time sequence. It therefore represents a sample time series of a stochastic simulation that needs to be repeated many times allowing for a statistical evaluation of the process.

Recently [?] have developed such an agent based model for the main features of the market for the computer chips used by High-End Gamers. They showed that the sales curves of 19 chips for two manufacturers (INTEL and AMD) over about 3 years can be qualitatively and quantitatively approximated using two main parameters representing the agents’ income allocated to the gaming hobby and the gamers motivation to improve their computer hardware.

Using the same data set, [?] have developed a population-growth model characterizing multiple product generations. Using an iterative-descent method they obtained parameter estimates for the population model.

This paper will develop a third approach, complementing the other two, by developing a kinetic model and subsequent partial differential equations for the agent based simulations. In doing so, we will use the High-End-Gamers data and simulation model as an inspiration for a proof of concept for kinetic marketing models. Following the approach that leads from the kinetic Boltzmann equations to hydrodynamic transport equations, the resulting partial differential equations will be transport equations for the number density of agents owning a particular chip moving in the conceptual spaces of affinity and ability to buy. We will show that the kinetic model captures many features ascribed to generic behavior of markets. We will discuss the distinct advantages of kinetic models over individualized agent based models and over heuristic aggregate compartmental models like in population dynamics:

- The resulting transport equations allow a scaling analysis identifying different market processes at different time scales.
- Such a scaling analysis generates a small number of dimensionless parameters that characterize the market dynamics. These dimensionless parameters are built from the larger number of original parameters of the agent based simulation.
- A kinetic model allows to analyze limiting behavior which are hard to simulate via agents validating intuition.
- Since the resulting transport equations are already based on averages, there is no need for multiple simulations, making those simulations fast and hence amenable to “what if” scenario explorations.

2. The agent based simulation model. We briefly review the setup for the agent based simulations for the High-End-Gamers market. More details can be found in [?].

There are only two manufacturers of these high-end computer chips worldwide. From January 2006 until April 2009 the Blue Team (INTEL) released 10 high-end chips and the Green Team (AMD) released 9 high-end chips. The estimated total monthly sales data of these chips are shown in Figure 1 for each company. All sales data are relative and displayed in arbitrary units.
The simulation involved a population of one to two thousand agents each playing 200 one-on-one games per month with random pairing against other agents. Games are reduced to a comparison of the performance of the current chip that an agent owns; the agent with the higher-performance chip wins with a high probability. Agents will keep a tally of their losses and are characterized by a loss threshold. Once the losses exceed the threshold, the agent decides that he/she needs a new chip and will buy the fastest chip that is on the market that she can afford. The simulation is non-autonomous since it is driven by the times that a particular chip with a particular performance and price will enter and exit the market. Those data were provided by INTEL and cover the full simulation time span of 40 months.

Notice that threshold variation will cover different skill levels for the agents: A top dog gamer presumably has a low threshold and few losses are enough to trigger the search for a new chip, whereas a newcomer to the game expects to lose and therefore has a larger threshold.

Figure 2 shows a typical result comparing the agent simulations to the sales data of multiple generations of products.

3. The kinetic model.

3.1. Playing the game. We consider agents with the following attributes:

- An agent owns a chip. The generations of chips are labeled by \( n = 1 \ldots N \).
- The number of losses that an agent as accumulated with the current chip is denoted by \( x \). We assume \( x \in \mathbb{R} \).
- The loss threshold variable for an agent is \( g \).
- The accumulated capital of the agent is called \( \mu \) and the rate of income for an agent is given by \( \gamma \). The state variable for the start of a kinetic model is \( F(X, G, M, N, t) \) the density of agents 1 and 2 have lost \( X = (x_1, x_2) \) games.

In addition there are a few more parameters that are relevant: A processor of type \( n \) has a cost of \( c_n \) and \( q \) is the frequency of games played (unit= \( \text{games/time} \)).

We define the two player density \( F(X, M, \Gamma, G, N, t) \) as the density of agents 1 and 2 at time \( t \) that have lost \( X = (x_1, x_2) \) games with \( M, \Gamma, G, N \) the two-vectors of capital, income rate, loss threshold, and chip numbers for the two players, respectively. A game is characterized by the fact that one player will win and another one
will lose. Only the loser will keep track of her loss and increase the appropriate $x$ value:

$$x_1 \rightarrow x_1 + r(N) q \Delta t, \quad x_2 \rightarrow x_2 + (1 - r(N)) q \Delta t,$$

$$P[r(N) = 1] = w(N), \quad P[r(N) = 0] = 1 - w(N).$$

where $w(n_1, n_2)$ is the probability that in a game between $n_1$ and $n_2$ the player with $n_2$ wins. Note: $w + w^T = \mathbf{1}^T$ has to hold.

In the same time $\Delta t$ both players increase their capital by the rate of income

$$\mu_j \rightarrow \mu_j + \Delta t \gamma_j, \quad j = 1, 2$$

Hence the evolution of the two agent density in the limit of $\Delta t \rightarrow 0$ becomes

$$\partial_t F = -w(N) q \partial_{x_1} F - [1 - w(N)] q \partial_{x_2} F - \Gamma \cdot \nabla_M F.$$

Making the usual assumptions about identical and statistically independent agents, factoring the two agent density as

$$F(X, M, G, N, t) = \prod_{j=1}^2 f(x_j, \mu_j, \gamma_j, g_j, n_j, t),$$
we can derive the time evolution for the single agent density
\[
\partial_t f(x, \mu, \gamma, g, n, t) + \partial_x \phi_x + \gamma \partial_\mu f = 0,
\]
\[
\phi_x(x, \mu, \gamma, g, n, t) = v_p(n, t) f(x, \mu, \gamma, g, n, t),
\]
\[
v_p(n, t) = \frac{q}{f(0)} \sum_m w(n, m) \rho(m, t),
\]
\[
\rho(n, t) = \int f(x, \mu, \gamma, g, n, t) \, dx \mu \gamma g
\]
\[
f(0) = \sum_n \rho(n, 0).
\]

Where \( \rho(t, n) \) is the number of agents at time \( t \) owning chip \( n \), \( \bar{f}(0) \) is the total number of agents in the system and hence \( \frac{1}{\bar{f}(0)} \sum_m w(n, m) \rho(m, t) \) is the fraction of all the games for a player that has chip number \( n \) that lead to a loss.

3.2. Buying a new chip. Rewriting the number density in terms of games lost, relative to thresholds we set
\[
f(x, \mu, \gamma, g, n, t) \rightarrow \frac{1}{g} f\left(\frac{x}{g}, \mu, \gamma, g, n, t\right), \quad v_p \rightarrow gv_p
\]
giving
\[
\partial_t f(x, \mu, \gamma, g, n, t) + \partial_x [v_p f] + \gamma \partial_\mu f = 0,
\]
\[
v_p(n, g, t) = \frac{q}{gf(0)} \sum_m w(n, m) \rho(m, t).
\]

Hence in the scaled \( x \) variable, an agent whose loss tally exceeds his threshold is at \( x > 1 \). Such agents continue to play but they are not keeping track of their losses any more. The number of those agents are described by
\[
u(\mu, \gamma, g, n, t) = \int_{-1}^{\infty} f(x, \mu, \gamma, g, n, t) \, dx.
\]

Once an agent decides to buy a new chip, a necessary condition for buying is that the available capital \( \mu \) is higher than the sales price of a chip that is better than her current chip. Once the two necessary conditions are met (\( x > 1, \mu > c_m \)) for some chip \( m > n \) we assume that the agent buys one of the possible chips. Hence the only agents that go into the waiting bin are those that do not have enough money to buy at that moment. Hence the time evolution for the waiting bin becomes
\[
\partial_t u(\mu, \gamma, g, n, t) + \gamma \partial_\mu u = \left[ 1 - H(\mu - c_{n+1}) \right] \phi_{p, \text{bin}}(\mu, \gamma, g, n, t)
\]
\[
\phi_{p, \text{bin}}(\mu, \gamma, g, n, t) = v_p(n, t) f(x = 1, \mu, \gamma, g, n, t).
\]

Buying instantly. For the players that have enough money to buy one or more chips, we define \( B(\mu, m, n) \) to be the probability that, upon reaching the threshold, the player upgrades from \( n \) to \( m \). That probability clearly depends on the current capital level \( \mu \). Once an agent upgrades, we deduct the cost of the chip from her capital and set the loss tally to zero. Hence that part of the outflux of the \( n \)-level transport equation becomes an influx at \( x = 0 \) in the level \( m \) transport equation. Equivalently, the influx at \( x = 0 \) at the \( n_{th} \) level is the sum of all the outfluxes that had at least \( \mu = c_n \) capital available. Hence
\[
v_p f(x = 0, \mu, \gamma, g, n, t) = \sum_{m=1}^{n-1} B(\mu + c_n, n, m) \phi_{p, \text{bin}}(\mu + c_n, \gamma, g, m, t).
\]
Buying after a waiting period. Once an agent in the waiting bin \( n \) reaches a capital \( c_m \), she decides to upgrade to level \( m \) with probability \( A(m, n) \). This leads to a loss term for the \( n^{th} \) level bin density \( u \) of the form

\[
- \sum_{m=1}^{N} \delta(\mu - c_m)A(m, n)\gamma u(\mu, \gamma, g, n, t)
\]

and an influx for the \( n^{th} \) level density \( f \) at the boundary corner \( x = 0 \) and \( \mu = 0 \) of the form

\[
\gamma f(x = 0, \mu = 0, \gamma, g, n, t) = \sum_{m=1}^{n-1} A(n, m)\gamma u(c_n, \gamma, g, m, t).
\]

To complete the boundary conditions we have no flux boundaries at \( \mu = 0 \) for \( f \) and \( u \): 
\[\gamma f(x, \mu = 0, \gamma, g, n, t) = 0, \quad \gamma u(\mu = 0, \gamma, g, n, t) = 0.\]

We assume that there is a maximal capital level \( \mu = M \geq c_N \) that a gamer might allocate to the game. Once a player reaches \( M \) the capital influx rate \( \gamma \) is set to zero. Setting \( A(N, n) = 1, \forall n \), we do not have to worry about cutting off in the bin equation, since the term \( \delta(\mu - c_N)A(N, n)\gamma u(c_N, \gamma, g, n, t) \) removes all agents, and therefore \( u(\mu, \gamma, g, n, t) = 0 \) for \( \mu > c_N \) holds automatically.

We summarize the equations for the evolution of the number density for the gamers who are happy with their chips and the number density for the gamers who are unhappy and are waiting for their capital to increase such that they can buy a new chip:

**Evolution equation**

\[
\begin{align*}
\partial_t f(x, \mu, \gamma, g, n, t) &+ \partial_x \phi_x + \gamma \partial_\mu f = 0, \\
\phi_x(x, \mu, \gamma, g, n, t) &= v_p(n, t)f(x, \mu, \gamma, g, n, t), \\
v_p(n, t) &= \frac{q}{f(0)} \sum_m n \rho(n, m)\rho(m, t), \\
\rho(n, t) &= \int f(x, \mu, \gamma, g, n, t) \, dx \mu \gamma + \int u(\mu, \gamma, g, n, t) \, d\mu \gamma.
\end{align*}
\]

**Boundary conditions**

\[
\begin{align*}
\gamma f(x, \mu = 0, \gamma, g, n, t) &= 0, \\
v_p f(x = 0, \mu, \gamma, g, n, t) &= \sum_{m=1}^{n-1} B(m + c_n, n, m)\phi_p.bin(\mu + c(n), \gamma, g, m, t), \\
\gamma f(x = 0, \mu = 0, \gamma, g, n, t) &= \sum_{m=1}^{n-1} A(n, m)\gamma u(c_n, \gamma, g, m, t).
\end{align*}
\]

**Waiting Bins**

\[
\begin{align*}
\partial_t u(\mu, \gamma, g, n, t) + \gamma \partial_\mu u &= \phi_{p.bin}(\mu, \gamma, g, n, t)[1 - \sum_{m=n+1}^{N} B(\mu, m, n)] \\
&- \sum_{m=n+1}^{N} \delta(\mu - c_m)A(m, n)\gamma u(\mu, \gamma, g, n, t), \\
\phi_{p.bin}(\mu, \gamma, g, n, t) &= v_p(n, t)f(x = 1, \mu, \gamma, g, n, t), \\
\gamma u(\mu = 0, \gamma, g, n, t) &= 0.
\end{align*}
\]

4. Scaling Analysis.

4.1. Dimensionless equations. We can scale all quantities to make the equations dimensionless: Scaling the losses \( x \) with the loss threshold \( g \), the capital \( \mu \) with the maximally accumulated capital \( M \), the monthly income \( \gamma \) by the maximal monthly income \( \Gamma \), the individual loss threshold \( g \) by the maximal loss threshold \( G \), the number density \( \rho \) by the maximal number of potential buyers \( P \), the velocity in loss space \( v_p \) by the frequency \( q \) of games played and time \( t \) by a scale \( T \), we transform
equations 3,4,5 into a dimensionless form:

**Dimensionless evolution equation**

\[
\begin{align*}
\partial_t f(x, \mu, \gamma, g, n, t) + \frac{\partial}{\partial x} \phi_x + \frac{\gamma T}{M} \partial_\mu \gamma f &= 0, \\
\phi_x(x, \mu, \gamma, g, n, t) &= \nu_p(n, t) f(x, \mu, \gamma, g, n, t), \\
\nu_p(n, t) &= \frac{1}{\gamma f(0)} \sum m w(n, m) \rho(m, t), \\
\rho(n, t) &= \int f(x, \mu, \gamma, g, n, t) \; dx \mu \gamma g + \int u(\mu, \gamma, g, n, t) \; d\mu \gamma g.
\end{align*}
\]

(6)

**Dimensionless boundary conditions**

\[
\begin{align*}
\gamma f(x, \mu = 0, \gamma, g, n, t) &= 0, \\
\nu_p f(x = 0, \mu, \gamma, g, n, t) &= \sum_{m=1}^{n-1} B(\mu + \frac{c_m}{M}, n, m) \phi_{p, \text{bin}}(\mu + \frac{c_m}{M}, \gamma, g, m, t), \\
\gamma f(x = 0, \mu = 0, \gamma, g, n, t) &= \sum_{m=1}^{n-1} A(n, m) \gamma u(\frac{c_m}{M}, \gamma, g, m, t).
\end{align*}
\]

(7)

**Dimensionless waiting bins**

\[
\begin{align*}
\partial_t u(\mu, \gamma, g, n, t) + \frac{\gamma T}{M} \partial_\mu u &= \frac{\phi_{p, \text{bin}}(\mu, \gamma, g, n, t)}{\gamma f(0)} \left[ 1 - \sum_{m=n+1}^{N} B(\mu, m, n) \right] \\
&- \frac{\gamma T}{M} \sum_{m=n+1}^{N} \delta(\mu - \frac{c_m}{M}) A(n, m) \gamma u(\mu, \gamma, g, n, t), \\
\phi_{p, \text{bin}}(\mu, \gamma, g, n, t) &= \nu_p(n, t) f(x = 1, \mu, \gamma, g, n, t), \\
\gamma u(\mu = 0, \gamma, g, n, t) &= 0.
\end{align*}
\]

(8)

4.2. Asymptotics. There are two possible time scales that govern the market dynamics: The financial scale \(T_F = \frac{M}{\Gamma} \) is the mean time it takes the monthly income to accumulate to the maximal capital allocated to gaming. This time scale is of the same order of magnitude as \(T_T = \frac{G}{\mu} \) which represents the mean time that it takes for the monthly income to reach the average cost of a chip.

We call the second time scale the gaming scale \(T_G = \frac{G}{\mu} \) which is the mean time it takes a player to accumulate enough losses to trigger the buying threshold, given a frequency \(q \) of playing games. We define \(\epsilon \) to be the dimensionless ratio

\[
\epsilon = \frac{T_G}{T_F} = \frac{\Gamma G}{M q}
\]

Case 1: Consider \(\epsilon << 1 \), i.e. \(T_G << T_F \) and measure time on the fast time scale \(T = T_G \). Hence we have a high frequency of reaching the loss threshold relative to the frequency of replenishing the capital. Slow financial recovery may have two sources: not enough income or high processor cost. A high frequency of reaching the loss threshold also has two sources: High frequency of games or a low loss threshold. Hence this case is characteristic for the highly committed and proficient gamer who spends a lot of time on gaming and is very good at it.

In this case, for the gaming equation Eq. 3, transport in the \(x \) direction is much larger than transport in the \(\mu \) direction. As a result agents will reach \(x = 1 \) on a timescale of \(O(1) \) and become frustrated because their financial situation has not kept pace. In all of these cases the fast equation for \(\epsilon = 0 \) is given for the game evolution by

\[
\partial_t f(x, \mu, \gamma, g, n, t) + \partial_x \phi_x = 0
\]

indicating a transport in \(x \) only with constant speed. Let us assume then that \(A(n+1, n) = 1 \), i.e. all agents will buy the new chip once they have the money to do so. Hence they begin gaming again at \(x = 0, \mu = 0 \). As a result, the flux into the bin equation on the fast timescale is given by \(\phi_x(1, 0, \gamma, g, n, t) = \nu_p(n, t) f(1, 0, \gamma, g, n, t) \).
Since on the fast time scale capital does not increase, no gamer has enough money to buy a new chip on that timescale and hence $B(m, n) = 0$. Thus the bin equation on the fast time scale becomes a trivial ODE accumulating all the gamers that reach the loss threshold,

$$
\partial_t u(\mu, \gamma, g, n, t) = \phi_{p, \text{bin}}(\mu, \gamma, g, n, t).
$$

On the slower financial timescale in the bin equation there is a drift towards increasing budget until the budget has reached the cost of the next chip $c_{n+1}$ at which time everybody buys and resets the dynamics again at the origin.

Hence, the fast timescale acts as a instantaneous reset map, moving agents from the point $x = 1, \mu = c_{n+1}$ to $x = 0, \mu = 0$. Thus the whole dynamics reduces to periodic buying behavior with a buying period of $T_{buy} = \frac{c_{n+1}}{\gamma}$. If $\gamma$ is constant every agent has the same period and the sales distribution is given by the initial distribution of capital.

**Case 2:** Consider $\varepsilon \gg 1$, i.e. $T_G >> T_F$ and measure time on the fast time scale $T = T_F$. Hence we have a low frequency of reaching the loss threshold relative to the frequency of replenishing the capital. Short $T_F$ can be generated by high income or low chip prices. Long $T_G$ may come from infrequent gaming or a high loss tolerance both are characteristic for a casual gamer.

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