The existence of metrics of nonpositive curvature on the Brady-Krammer complexes for finite-type Artin groups

Woonjung Choi
Texas A&M University

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1 Overview

Because almost all questions regarding finitely presented groups are known to be undecidable in general, researchers of infinite groups have long focused on those special classes of groups where everything seems to work: free groups and their automorphisms, surface groups, one-relator groups, Coxeter groups, Artin groups, groups which satisfy some sort of small-cancellation conditions, etc. Beginning in the early 1980s Misha Gromov strongly advocated the viewpoint that every finitely generated group has an intrinsic geometry which has a major impact on its algebraic structure, and over the past twenty years it has become clear that many of the standard classes of groups are negatively or non-positively curved in a precise geometric sense. Thus non-positive curvature is now seen as a partial explanation of the exceptional nature of these classes of groups.

Building on Gromov’s work (and the work of other differential geometers) many theories of curvature have been developed which are closely related to infinite group theory. Because there are many different competing theories, there has been a major effort in recent years to see which theories of curvature can be applied to each of the standard classes of groups studied by geometric group theorists and, in particular, to see if the various theories can be distinguished based on these investigations.

The focus of my dissertation will be on one of the standard classes of groups (Artin groups of finite-type) and one of the standard theories of non-positive curvature as developed by Cartan, Alexandrov and Topogonov (CAT(0) spaces and groups, named in honor of these three early researchers in the area). A standard reference for this theory is the book by Bridson and Haefliger [3]. Both the groups involved and the theory being applied will be defined in detail below.

2 Background

My dissertation focuses on the existence (or more specifically non-existence) of metrics of a particular type on some simplicial complexes of interest to geometric group theorists. I will begin by introducing

1. the groups we’re interested in
2. the type of metrics we’re considering
3. the specific simplicial complexes we’re investigating

Then in Section 3, I’ll discuss previously known results and in Section 4, I’ll outline the approaches to the proof of my main problem.

2.1 The groups

Coxeter groups and Artin groups are classes of groups defined by finite presentations of a very special form. This general form is derived from the presentations for all finite reflection groups (in the case of Coxeter groups) and from the braid groups (for Artin groups). Coxeter groups and Artin groups are well-behaved enough to prove strong results, but complicated enough to produce numerous counterexamples to long-standing conjectures. As a result, they are an important testing ground for the various general theories of nonpositive curvature within geometric group theory.

Let $\Gamma$ be a finite graph with edges labeled by integers greater than 1, and let $(a, b)^n$ be the first $n$ letters of $(ab)^n$.

**Definition** The Artin group $A_\Gamma$ is the group generated by a set in one to one correspondence with the vertices in $\Gamma$ with a relation of the form $(a, b)^n = (b, a)^n$ whenever the vertices corresponding to $a$ and $b$ are joined by an edge labeled $n$.

**Definition** The Coxeter group $W_\Gamma$ is the Artin group $A_\Gamma$ modulo the relations $a^2 = 1$ for each generator $a$.

A simple example is shown in Fig. 1.

The finite Coxeter groups have been classified. An Artin group defined by the same labeled graph as a finite Coxeter is called a finite-type Artin group.

2.2 The theory

The particular theory of curvature I will be using is the theory of CAT(0) spaces.

**Definition** Let $(X, d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or more briefly a geodesic from $x$ to $y$) is a map $c$ from a closed
Artin presentation:
\[ \langle a, b, c \mid aba = bab, ac = ca, bc = bc \rangle \]

Coxeter presentation:
\[ \left\langle a, b, c \mid \begin{array}{l} aba = bab, ac = ca, \\ bc = cb \end{array} \right\rangle \]

Figure 1: An example for Coxeter group and Artin group
Figure 2: A geodesic triangle and the comparison triangle

to define a space on which these groups act with a metric that we can show satisfies the CAT(0) condition. We should note that even once the space with a metric has been defined it is nontrivial to prove that the CAT(0) condition is satisfied.

The easiest way to construct a CAT(0) space is by adding metrics to simplicial complexes. The metrics we will use are piecewise Euclidean.

**Definition**¹ A piecewise Euclidean (PE) complex $X$ is a simplicial complex in which each simplex is given a Euclidean metric and the induced metrics on the intersections always agree.

Determining whether a PE complex is nonpositively curved can be reformulated as a condition on the links of cells.

**Definition** Let $X$ be a PE complex and $x$ be a point in $X$. The *link of $x$ in $X$* is the set of unit tangent vectors to $X$ at $x$. When $x$ lies in the interior of a cell $B$ of $X$, the *link of $B in $X* is the set of unit tangent vectors to $x$ in $X$ which are orthogonal to $B$.

**Theorem.** A PE complex is nonpositively curved iff the link of each cell does not contain a closed geodesic loop of length less than $2\pi$.

In high dimensions, testing for closed, short geodesic loops is complicated, but in low dimensions Murray Elder and Jon McCammond have developed procedures to carry out this type of test.

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¹See [4] for notations and more detail.
2.3 The spaces

For each finite-type Artin group there is a nice concrete space to work with called the Brady-Krammer complex. Brady-Krammer complexes are simplicial complexes constructed recently by Tom Brady and Daan Krammer on whose universal cover the braid groups and other finite-type Artin groups act properly cocompactly by isometries. The complex for a finite-type Artin group can be constructed as follows.

- Consider a finite Coxeter group \((G, S)\) where \(S\) is the standard generating set. It can also be generated by using the set of all reflections \(T\) where a reflection is defined as a conjugate of a standard generator. Notice that \(S\) is a subset of \(T\).

- Create a new Cayley graph with respect to \(T\).

- Orient the graph based on \(l_T\) (length function) and get a poset where \(l_T(x) := \) the least length from 1 in the Cayley graph.

- Recall that a \textit{coxeter element} is the product of all the standard generators in any order. Pick a coxeter element and restrict the poset to the interval between the coxeter element and the identity. Then the interval, \(P_G\), is the poset of factorizations of the coxeter element into reflections. It has been shown that this poset is a graded, bounded, self-dual lattice. In the special case where \(G\) is the symmetric group \(S_n\), \(P_G\) is the noncrossing partition lattice \(NC_n\).

- Realize this poset as a simplicial complex and label the edges with group elements. In particular, every edge in the simplicial complex connects vertices which are labeled by elements of the Coxeter group, say \(g\) and \(h\), oriented from \(g\) to \(h\) if \(g < h\) in the poset. The edge label in this case is \(g^{-1}h\).

- Finally glue simplices according to their labels. In other words, if two simplices have identical labels on their oriented edges, we identify them.

T. Brady and D. Krammer proved independently that carrying out this procedure for the symmetric group creates an Eilenberg-Maclane space for the corresponding braid group. The extension to other finite-type Artin groups was established by T. Brady - C. Watt [2] and D. Bessis [1], independently.
Example

\( (A_2, \{a, b\}) \)

\[ \text{cayley graph of } (G, T) \text{ graded by distance from } 1 \]

\( P_G \)

\( K \)

\( K' \)

\( \delta = \text{Coxeter element} = \prod_{s \in S} s \)

Figure 3: Brady-Krammer Complexes
3 Problem Statement

My dissertation focuses on the existence of non-positively curved metrics on the Brady-Krammer complexes for particular finite-type Artin groups. The existence of such a metric would prove that these groups are CAT(0) groups. Some positive results already exist.

**Theorem (T.Brady-J.McCammond).** The finite-type Artin groups with at most 3 generators are CAT(0) groups, and the Artin groups $A_4$ and $B_4$ are CAT(0) groups.

The proof finds a nonpositively curved PE metric on the appropriate Brady-Krammer complex. At this point it is natural to conjecture the following.

**Conjecture.** The Brady-Krammer complex for each Artin group of finite type supports a nonpositively curved PE metric.

I plan to investigate the existence of such a metric for the four generator finite type Artin groups. The fact that the dimension of the Brady-Krammer complexes for these groups is small (specifically dimension 4) makes this feasible. Either answer would be of interest. A positive answer would provide further evidence for the conjecture and extend our ability to construct CAT(0) spaces while a negative answer would disprove the conjecture and show that Bestvina’s asymmetric version of nonpositive curvature is strictly weaker than the standard version.

4 Approach

Consider the group action on the universal cover of the Brady-Krammer complex for either $D_4$ or $F_4$. Since each group has a finite index subgroup which is a direct product of an infinite cyclic group with another subgroup, by the splitting theorem, in order for the universal cover of the Brady-Krammer complex to be CAT(0), it must be a direct product of a three dimensional CAT(0) complex with the real line. The simplices in the universal cover naturally decompose into columns which must metrically be a Euclidean tetrahedron cross the real line. In particular, we can construct the simplicial structure of this three dimensional cross section. The universal cover will support a CAT(0) metric if and only if we can find a CAT(0) metric on
this three dimensional cross section. I will first examine which piecewise Euclidean metrics on the three dimensional cross-section complex have dihedral angles which make the links of the edges (which are finite graphs) large in the sense that they do not contain closed geodesic loops of length less than $2\pi$. Once I have determined the class of metrics which meet these requirements, I will then examine the link of the vertex (which is unique up to translation).

Because of the size of the posets and complexes involved, I plan to use a computer program to help analyze these posets and spaces.

The program `coxeter.g` is a set of GAP routines used to examine Brady-Kramer complexes. It was initially developed to test the curvature of the Brady-Kramermer complexes using a “natural” metric and I am extensively reworking the routines so that they

- find the poset
- find the simplicial structure of the three dimensional cross-section
- find representative vertices and edges up to automorphism
- find the graphs for the edge links
- find the simple cycles in these graphs
- find the linear system of inequalities which needs to be satisfied by the dihedral angles of the tetrahedra.

The linear systems for $D_4$ and $F_4$ should be small enough to analyze by hand because the defining graphs have some symmetries. Once this analysis is complete I will proceed to the study of the vertex link. The remaining four generator Artin group of finite type is called $H_4$. The argument for $H_4$ would proceed along the same lines except that the linear system for $H_4$ is probably not small enough to analyze by hand since its defining graph has no symmetries. Therefore, the case of $H_4$ will be a subject of further research after the dissertation.

**References**

